

Q1a $2xy - 9y^2 + 9 = 0$.

Implicit differentiation: $2y + 2x \frac{dy}{dx} - 18y \frac{dy}{dx} = 0$,

$\therefore y + x \frac{dy}{dx} - 9y \frac{dy}{dx} = 0$, $(9y - x) \frac{dy}{dx} = y$,

$\therefore \frac{dy}{dx} = \frac{y}{9y - x}$.

Q1b When $y = 1$, $x = 0$, $\therefore \frac{dy}{dx} = \frac{1}{9}$.

Q2 $\frac{dy}{dx} = x\sqrt{x^2 - 16}$, $\therefore y = \int x\sqrt{x^2 - 16} dx$.

Let $u = x^2 - 16$, $\frac{du}{dx} = 2x$, $x = \frac{1}{2} \frac{du}{dx}$.

$\therefore y = \int \frac{\sqrt{u}}{2} \frac{du}{dx} dx = \int \frac{\sqrt{u}}{2} du = \frac{u^{\frac{3}{2}}}{\frac{2 \times \frac{3}{2}}{2}} + C = \frac{1}{3}(x^2 - 16)^{\frac{3}{2}} + C$.

Given $y = 13$ when $x = 5$, $\therefore C = 4$.

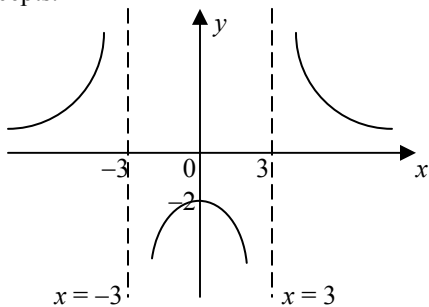
$\therefore y = \frac{1}{3}(x^2 - 16)^{\frac{3}{2}} + 4$.

Q3a $y = \frac{36}{2x^2 - 18} = \frac{18}{x^2 - 9} = \frac{18}{(x-3)(x+3)}$

Asymptotes are $x = -3$ and $x = 3$.

y-intercept: Let $x = 0$, $\therefore y = -2$.

No x-intercepts.



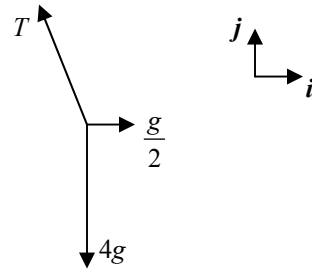
Q3b Partial fractions: $\frac{18}{(x-3)(x+3)} = \frac{3}{x-3} - \frac{3}{x+3}$.

Area = $-2 \times \int_0^2 \left(\frac{3}{x-3} - \frac{3}{x+3} \right) dx = -6 \int_0^2 \left(\frac{1}{x-3} - \frac{1}{x+3} \right) dx$

$= -6 \left[\log_e |x-3| - \log_e |x+3| \right]_0^2 = -6 \left[\log_e \frac{|x-3|}{|x+3|} \right]_0^2$

$= -6 \log_e \frac{|-1|}{|5|} = -6 \log_e \left(\frac{1}{5} \right) = 6 \log_e 5$.

Q4a



Q4b $R = 0$

$T + \frac{g}{2}i - 4gj = 0$, $\therefore T = -\frac{g}{2}i + 4gj$,

$\therefore T = \sqrt{\left(-\frac{g}{2}\right)^2 + (4g)^2} = g\sqrt{\frac{1}{4} + 16} = g\sqrt{\frac{65}{4}} = \frac{g\sqrt{65}}{2}$.

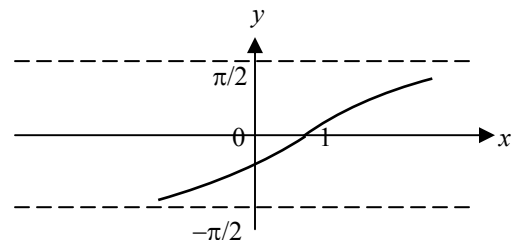
Q5a $\tan\left(\frac{\pi}{4}\right) = \tan\left(2 \times \frac{\pi}{8}\right) = \frac{2 \tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$,

$\therefore 1 = \frac{2 \tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$, $1 - \tan^2\left(\frac{\pi}{8}\right) = 2 \tan\left(\frac{\pi}{8}\right)$,

$\therefore \tan^2\left(\frac{\pi}{8}\right) + 2 \tan\left(\frac{\pi}{8}\right) - 1 = 0$, a quadratic equation.

$\therefore \tan\left(\frac{\pi}{8}\right) = \frac{-2 + \sqrt{2^2 - 4(1)(-1)}}{2(1)} = \sqrt{2} - 1$, since $\tan\left(\frac{\pi}{8}\right) > 0$.

Q5b The graph $y = \tan^{-1}(x-1)$ is shown below.

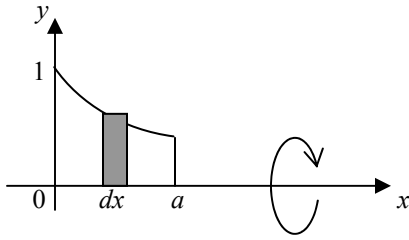


$\therefore \tan^{-1}(x-1) + a \tan\left(\frac{\pi}{8}\right) > 0$ for all x if $a \tan\left(\frac{\pi}{8}\right) \geq \frac{\pi}{2}$,

i.e. $a(\sqrt{2} - 1) \geq \frac{\pi}{2}$, $\therefore a \geq \frac{\pi}{2(\sqrt{2} - 1)}$.

Hence minimum value of $a = \frac{\pi}{2(\sqrt{2} - 1)} = \frac{\pi(\sqrt{2} + 1)}{2}$.

Q6a



$$V = \int_0^a \pi y^2 dx = \int_0^a \pi (e^{-x})^2 dx = \int_0^a \pi e^{-2x} dx$$

$$Q6b \quad V = \left[\frac{\pi e^{-2x}}{-2} \right]_0^a = \left(\frac{\pi e^{-2a}}{-2} \right) - \left(\frac{\pi e^0}{-2} \right) = \frac{\pi}{2} (1 - e^{-2a})$$

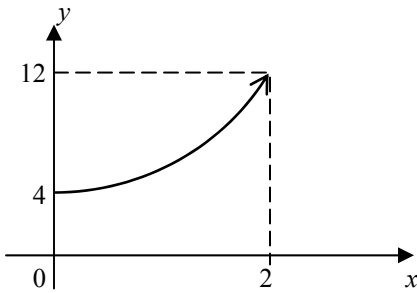
$$Q6c \quad \frac{\pi}{2} (1 - e^{-2a}) = \frac{5\pi}{18}, \quad 1 - e^{-2a} = \frac{5}{9}, \quad e^{-2a} = \frac{4}{9}, \quad e^{2a} = \frac{9}{4},$$

$$2a = \log_e \left(\frac{9}{4} \right), \quad a = \frac{1}{2} \log_e \left(\frac{9}{4} \right) = \log_e \left(\frac{3}{2} \right).$$

$$Q7a \quad x = \sqrt{t-2}, \quad y = 2t \text{ for } 2 \leq t \leq 6.$$

Eliminate t : $x^2 = t - 2$, $\therefore t = x^2 + 2$,
 $\therefore y = 2(x^2 + 2)$ for $0 \leq x \leq 2$ and $4 \leq y \leq 12$.

Q7b



$$Q8 \quad \int \frac{2+6x}{\sqrt{4-x^2}} dx = \int \frac{2}{\sqrt{4-x^2}} dx + \int \frac{6x}{\sqrt{4-x^2}} dx$$

$$= 2 \sin^{-1} \left(\frac{x}{2} \right) + \int \frac{-3}{\sqrt{u}} \frac{du}{dx} dx \quad (\text{where } u = 4 - x^2 \text{ and } \frac{du}{dx} = -2x)$$

$$= 2 \sin^{-1} \left(\frac{x}{2} \right) - 3 \int u^{-\frac{1}{2}} du$$

$$= 2 \sin^{-1} \left(\frac{x}{2} \right) - 6u^{\frac{1}{2}}$$

$$= 2 \sin^{-1} \left(\frac{x}{2} \right) - 6\sqrt{4-x^2}.$$

Q9a $1 + i\sqrt{3}$ is in the first quadrant of the complex plane.

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2, \quad \theta = \tan^{-1} \left(\frac{\sqrt{3}}{1} \right) = \frac{\pi}{3},$$

$$\therefore 1 + i\sqrt{3} = 2 \operatorname{cis} \left(\frac{\pi}{3} \right).$$

$$Q9b \quad z^2 + 2z - i\sqrt{3} = 0$$

Complete the square: $z^2 + 2z + 1 - 1 - i\sqrt{3} = 0$,

$$\therefore (z+1)^2 - (1 + i\sqrt{3}) = 0,$$

$$(z+1)^2 = 1 + i\sqrt{3}.$$

In polar form: $(z+1)^2 = 2 \operatorname{cis} \left(\frac{\pi}{3} + 2k\pi \right).$

$$\therefore z+1 = \sqrt{2} \operatorname{cis} \frac{1}{2} \left(\frac{\pi}{3} + 2k\pi \right).$$

Let $k = 0$, $z+1 = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{6} \right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i$,

$$\therefore z = \left(\frac{\sqrt{6}-2}{2} \right) + \frac{\sqrt{2}}{2} i.$$

Let $k = -1$,

$$z+1 = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{6} - \pi \right) = \sqrt{2} \operatorname{cis} \left(-\frac{5\pi}{6} \right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i,$$

$$\therefore z = \left(\frac{-\sqrt{6}-2}{2} \right) - \frac{\sqrt{2}}{2} i.$$

Alternative method: Let $z = a + bi$ where $a, b \in \mathbb{R}$, be a

solution to the equation $z^2 + 2z - i\sqrt{3} = 0$.

$$\therefore (a + bi)^2 + 2(a + bi) - i\sqrt{3} = 0.$$

Expand and collect real parts and imaginary parts:

$$(a^2 + 2a - b^2) + (2ab + 2b - \sqrt{3})i = 0.$$

$$\therefore a^2 + 2a - b^2 = 0 \quad (1)$$

$$\text{and } 2ab + 2b - \sqrt{3} = 0 \quad (2)$$

$$\text{From (1), } a^2 + 2a = b^2 \quad (3)$$

$$\text{From (2), } 2b(a+1) = \sqrt{3}, \therefore 4b^2(a+1)^2 = 3,$$

$$\therefore 4b^2(a^2 + 2a + 1) = 3 \quad (4)$$

$$\text{Substitute (3) in (4), } 4b^2(b^2 + 1) = 3, \therefore 4(b^2)^2 + 4(b^2) - 3 = 0,$$

$$\therefore (2b^2 + 3)(2b^2 - 1) = 0.$$

$$\text{Since } 2b^2 + 3 > 0, \therefore 2b^2 - 1 = 0, \therefore b = \pm \frac{\sqrt{2}}{2}.$$

$$2b(a+1) = \sqrt{3}, \therefore a = \frac{\sqrt{3}}{2b} - 1 = \pm \frac{\sqrt{3}}{2} - 1 = \frac{\pm\sqrt{6}-2}{2}.$$

Hence the solutions are:

$$z = \left(\frac{\sqrt{6}-2}{2} \right) + \frac{\sqrt{2}}{2} i \text{ or } z = \left(\frac{-\sqrt{6}-2}{2} \right) - \frac{\sqrt{2}}{2} i.$$

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