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2006 VCAA Specialist Maths Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
C	E	D	C	B	B	E	A	C	B	C

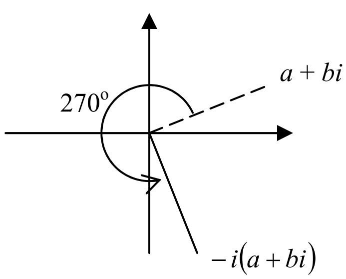
12	13	14	15	16	17	18	19	20	21	22
D	E	B	D	D	B	A	A	D	A	E

Q1 $(y-1)=\pm\frac{2}{3}(x-2)$, $\therefore 3y-3=2x-4$, i.e. $2x-3y=1$ or $3y-3=-2x+4$, i.e. $2x+3y=7$

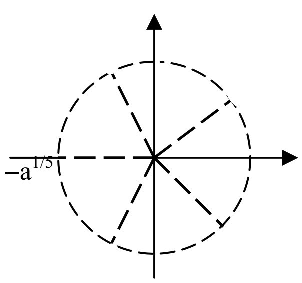
Q2 Consider $y=(x-4)(x+2)$, turning point at $(1,-9)$. For $f(x)=\frac{1}{y}$, turning point at $\left(1,-\frac{1}{9}\right)$. Asymptotes: $x=4$, $x=-2$.

Q3 $x=1+t$ (1), $y=1-t$ (2)
(1) + (2), $x+y=2$, $\therefore y=-x+2$

Q4 Since $-i(a+bi)$ can be expressed as $i^3(a+bi)$, $\therefore -i(a+bi)$ is the anticlockwise rotation of $a+bi$ about O by 270°.



Q5



Q6 Since a could be -1, $\therefore |z-(a+2i)|\le 1$.

Q7 Let $z=x+yi$.
$z+\bar{z}=0$, $\therefore x=0$.
$3\,\mathrm{Re}(z)=\mathrm{Im}(z)$, $\therefore 3x=y$
$z=iz$, $\therefore x=y$
$\mathrm{Re}(z)-2\,\mathrm{Im}(z)=0$, $\therefore x-2y=0$

$\mathrm{Re}(z)+\mathrm{Im}(z)=1$, $\therefore x+y=1$ and it does not pass through the origin.

Q8 When $y=-3$, $x=2$.

Implicit diff. $6x^2-2y\frac{dy}{dx}=0$, $\therefore \frac{dy}{dx}=\frac{3x^2}{y}=-4$

Q9 $u=x^2+1$, $\frac{du}{dx}=2x$, $x=\frac{1}{2}\frac{du}{dx}$.
When $x=a$, $u=a^2+1$.
When $x=b$, $u=b^2+1$.
$\therefore \int_a^b x(x^2+1)^5\,dx=\frac{1}{2}\int_a^b u^5\frac{du}{dx}dx=\frac{1}{2}\int_{a^2+1}^{b^2+1}u^5\,du$.

Q10 x kg dissolved, $\therefore 8-x$ kg undissolved.
$\frac{dx}{dt}=\frac{5}{100}(8-x)$, $x<8$. $\therefore \frac{dx}{dt}=\frac{8-x}{20}$.

Q11 Ellipses of the form $\frac{x^2}{2}+y^2=C$.

Implicit diff., $x+2y\frac{dy}{dx}=0$, $\therefore \frac{dy}{dx}=-\frac{x}{2y}$.

Q12 $v=\sin 2x$, $\frac{dv}{dx}=2\cos 2x$,
$a=v\frac{dv}{dx}=(\sin 2x)(2\cos 2x)=\sin 4x$.

Q13 R is at $(2,3)$ at $t=6$. S is at $(2,3)$ at $t=4$. R and S are never in the same position.

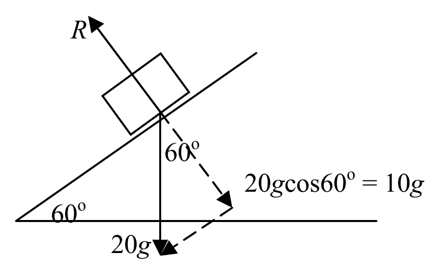
Q14 Velocity vector gives direction of motion.
$\boldsymbol{v}(t)=-\boldsymbol{i}-\frac{3}{\sqrt{t}}\boldsymbol{j}$, $\therefore$ $\boldsymbol{v}(9)=-\boldsymbol{i}-\frac{3}{\sqrt{9}}\boldsymbol{j}=-\boldsymbol{i}-\boldsymbol{j}$.

Q15 $\boldsymbol{a}=-\boldsymbol{c}$, $\therefore \boldsymbol{a}+\boldsymbol{c}=\boldsymbol{0}$

Q16 C and D are both perpendicular to the given vector (dot product = 0), but only D is a unit vector.

Q17 $\cos\theta=\boldsymbol{u}\bullet\boldsymbol{v}/(\mathrm{uv})=\frac{1}{\sqrt{2}}$, $\therefore \theta=45^0$.

Q18



$R=10g$

Q19

A. $12\cos 30^0 = 10.39$

B. $10\cos 10^0 = 9.85$

C. 10

D. $10\cos 20^0 = 9.40$

E. $12\cos 45^0 = 8.49$

Q20 Due to symmetry, $F_2 = F_3$.

$F_1 = F_2\cos 30^0 + F_3\cos 30^0 = \frac{\sqrt{3}}{2}F_2 + \frac{\sqrt{3}}{2}F_3 = \sqrt{3}F_2$,

$\therefore F_2 = F_3 = \frac{1}{\sqrt{3}}F_1 = \frac{\sqrt{3}}{3}F_1$.

Q21 $N = 8g\cos 30^0 = 4\sqrt{3}g$, $F = 8g\sin 30^0 = 4g$.

For equilibrium to be maintained, $F \le \mu N$, $\therefore \mu \ge \frac{F}{N}$,

$\therefore \mu \ge \frac{1}{\sqrt{3}}$.

Q22 $5g - T = 5a$ (1), $T - 2g = 2a$ (2).

(1) + (2), $3g = 7a$, $\therefore a = \frac{3g}{7}$.

SECTION 2

Q1a $V = \int_0^5 \pi\left(\frac{6x}{\sqrt{1+x^3}}\right)^2 dx = \int_0^5 \frac{36\pi x^2}{1+x^3}dx$

Q1b Let $u = 1 + x^3$, $\frac{du}{dx} = 3x^2$. When $x = 0$, $u = 1$. When $x = 5$, $u = 126$.

$V = \int_0^5 \frac{12\pi}{u}\frac{du}{dx}dx = \int_1^{126}\frac{12\pi}{u}du$.

Q1c $V = \left[12\pi\log_e u\right]_1^{126} = 182$ cm^3.

Q1d Given $\frac{dx}{dt} = +2$ cms^{-1}, $\frac{dy}{dt} = \frac{dy}{dx}\times\frac{dx}{dt} = \frac{2(6-3x^3)}{(1+x^3)^{\frac{3}{2}}}$.

Q1e $A = \pi y^2$, $\frac{dA}{dy} = 2\pi y = \frac{12\pi x}{(1+x^3)^{\frac{1}{2}}}$,

$\therefore \frac{dA}{dt} = \frac{dA}{dy}\times\frac{dy}{dt} = \frac{24\pi x(6-3x^3)}{(1+x^3)^2}$.

Q1f A is maximum when $\frac{dy}{dx} = 0$,

i.e. $\frac{6-3x^3}{(1+x^3)^{\frac{3}{2}}} = 0$, $\therefore x^3 = 2$, $x = \sqrt[3]{2}$ cm.

Q2a $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = 6\boldsymbol{i}$, $\overrightarrow{BD} = \boldsymbol{d} - \boldsymbol{b} = 10\boldsymbol{j}$, $\therefore \overrightarrow{AC} \perp \overrightarrow{BD}$.

Q2b $\overrightarrow{DA} = \boldsymbol{a} - \boldsymbol{d} = -3\boldsymbol{i} - 9\boldsymbol{j}$, $\overrightarrow{DC} = \boldsymbol{c} - \boldsymbol{d} = 3\boldsymbol{i} - 9\boldsymbol{j}$,

$\cos\angle ADC = \frac{\overrightarrow{DA}\bullet\overrightarrow{DC}}{\left|\overrightarrow{DA}\right|\left|\overrightarrow{DC}\right|} = \frac{72}{90} = 0.8$.

Q2c $\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -3\boldsymbol{i} + \boldsymbol{j}$, $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j}$,

$\cos\angle ABC = \frac{\overrightarrow{BA}\bullet\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{-8}{10} = -0.8$.

$\therefore \cos\angle ABC = -\cos\angle ADC = \cos(180^0 - \angle ADC)$,

$\therefore \angle ABC = 180^0 - \angle ADC$. Hence the two angles are supplementary.

Q2d $\overrightarrow{PA} = \boldsymbol{a} - \boldsymbol{p} = -3\boldsymbol{i} - 4\boldsymbol{j}$, $\overrightarrow{PC} = \boldsymbol{c} - \boldsymbol{p} = 3\boldsymbol{i} - 4\boldsymbol{j}$,

$\cos\angle APC = \frac{\overrightarrow{PA}\bullet\overrightarrow{PC}}{\left|\overrightarrow{PA}\right|\left|\overrightarrow{PC}\right|} = \frac{7}{25} = 0.28$.

$\cos 2\angle ADC = 2\cos^2\angle ADC - 1 = 2(0.8)^2 - 1 = 0.28$,

$\therefore \angle APC = 2\angle ADC$.

Q3ai $a = \frac{R}{m} = \frac{105600}{48000} = 2.2$ ms^{-2}.

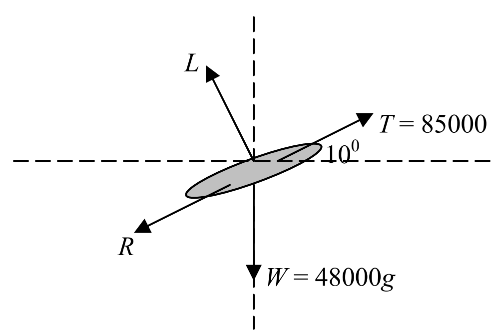
Q3aii $u = 0$, $v = 70$, $a = 2.2$, t? Use $v = u + at$,

$t = \frac{v-u}{a} = 31.8$ s.

Q3aiii $u = 0$, $v = 70$, $a = 2.2$, s? Use $v^2 = u^2 + 2as$,

$s = \frac{v^2 - u^2}{2a} = 1114$, distance = 1114 m.

Q3bi

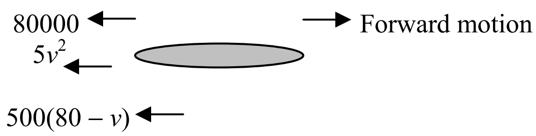


Q3bii $85000 - R - 48000g\sin 10^0 = 0$,

$L - 48000g\cos 10^0 = 0$

Q3biii $L = 48000g\cos 10^0 = 463254$ newtons.

Q3ci



$$a = \frac{F}{m} = \frac{-5v^2 - 500(80-v) - 80000}{48000},$$

$$\therefore a = -\frac{v^2 - 100v + 24000}{9600}.$$

Q3cii $v\frac{dv}{dx} = -\frac{v^2 - 100v + 24000}{9600}$,

$$\therefore \frac{dx}{dv} = -\frac{9600v}{v^2 - 100v + 24000}$$

When $x = 0$, $v = 80$.

At $v = 10$, $x = \int_{80}^{10} -\frac{9600v}{v^2 - 100v + 24000}\,dv$,

$$\therefore x = \int_{10}^{80} \frac{9600v}{v^2 - 100v + 24000}\,dv$$

Q3ciii By using graphics calculator, $x = 1385$ m.

Q4a $\frac{dy}{dt} = 1 - y$, $\frac{dt}{dy} = \frac{1}{1-y}$ where $t \ge 0$ and $y > 0$.

$t = \int \frac{1}{1-y}\,dy = -\log_e|1-y| + c$,

$\therefore \log_e|1-y| = c - t$. Clearly $y \ne 1$.

Q4b $\frac{dx}{dt} = \frac{1}{y}$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-y}{\frac{1}{y}} = y(1-y)$ where $y > 0$ and $y \ne 1$.

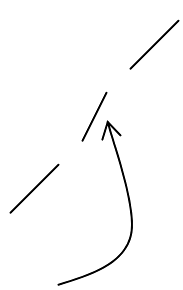
Q4ci $\frac{dy}{dx} = y - y^2$. By implicit diff.,

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - 2y\frac{dy}{dx} = (1-2y)\frac{dy}{dx} = (1-2y)y(1-y).$$

Q4cii To find the inflection point(s) for $0 < y < 1$, let $\frac{d^2y}{dx^2} = 0$.

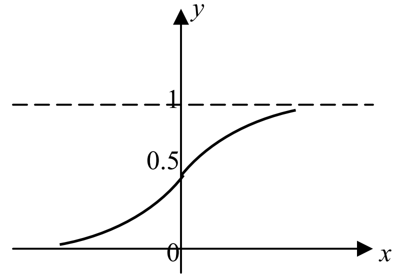
$\therefore (1-2y)y(1-y) = 0$, $\therefore y = \frac{1}{2}$.

y	$\frac{dy}{dx}$
$\frac{3}{4}$	$\frac{3}{16}$
$\frac{1}{2}$	$\frac{1}{4}$
$\frac{1}{4}$	$\frac{3}{16}$



Inflection point

Q4d



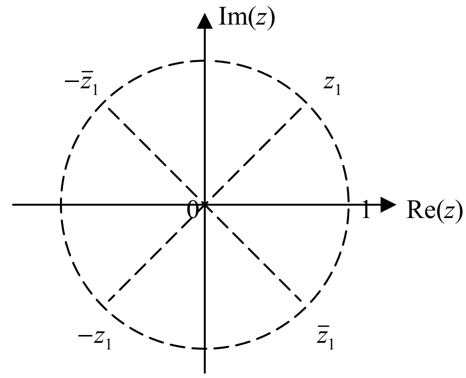
Q4e $\frac{dy}{dx} = y - y^2$, $y_{new} \approx y_{old} + h\left.\frac{dy}{dx}\right|_{y_{old}}$

$x = 0$, $y = 2$

$x = \frac{1}{4}$, $y \approx 2 + \frac{1}{4}(2 - 2^2) = \frac{3}{2}$

$x = \frac{1}{2}$, $y \approx \frac{3}{2} + \frac{1}{4}\left(\frac{3}{2} - \left(\frac{3}{2}\right)^2\right) = \frac{21}{16}$.

Q5ai



Q5aii $|z - \bar{z}_1| = |z + \bar{z}_1|$

Q5b $\cos\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1$, $\therefore \frac{1}{\sqrt{2}} = 2\cos^2\left(\frac{\pi}{8}\right) - 1$,

$$\therefore \cos^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}+1}{2\sqrt{2}} = \frac{2+\sqrt{2}}{4},$$

$$\therefore \cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

$-\frac{\sqrt{2+\sqrt{2}}}{2}$ is rejected because $\frac{\pi}{8}$ is in the first quadrant, $\cos\left(\frac{\pi}{8}\right) > 0$.

Q5c $\sin^2\left(\frac{\pi}{8}\right) + \cos^2\left(\frac{\pi}{8}\right) = 1$,

$$\therefore \sin\left(\frac{\pi}{8}\right) = \sqrt{1 - \cos^2\left(\frac{\pi}{8}\right)} = \sqrt{1 - \frac{2+\sqrt{2}}{4}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Q5d $\left(\sqrt{\frac{2+\sqrt{2}}{4}}+\frac{\sqrt{2-\sqrt{2}}}{2}i\right)^7=\left(\cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right)\right)^7$

$=\cos\left(\frac{7\pi}{8}\right)+i\sin\left(\frac{7\pi}{8}\right)$.

Q5e $\left(\sqrt{\frac{2+\sqrt{2}}{4}}+\frac{\sqrt{2-\sqrt{2}}}{2}i\right)^n=\left(\cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right)\right)^n$

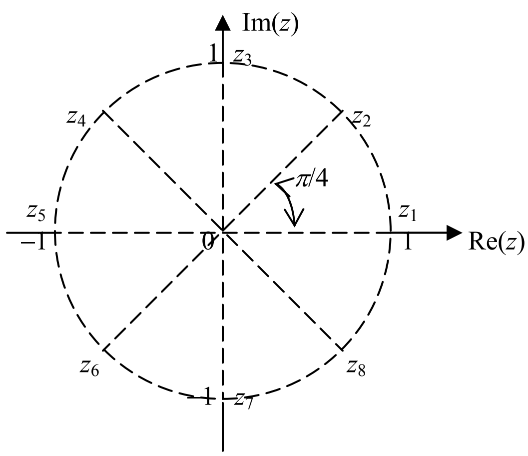
$=\cos\left(\frac{n\pi}{8}\right)+i\sin\left(\frac{n\pi}{8}\right)$.

$\left(\sqrt{\frac{2+\sqrt{2}}{4}}+\frac{\sqrt{2-\sqrt{2}}}{2}i\right)^n$ is a real number when $\sin\left(\frac{n\pi}{8}\right)=0$.

It occurs when $\frac{n\pi}{8}=k\pi$, i.e. $n=8k$ for $k=0,\pm1,\pm2,....$

$\therefore n=0,\pm8,\pm16,......$

Q5f



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