VCAA Mathematical Methods 3/4 Exam 2 Solutions 2005
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Q1ai $f(t)=2e^{-t}$, $t \geq 0$ is a decreasing function. At $t=0$, $y=f(0)=2$. As $t \to \infty$, $y \to 0^+$. $\therefore$ the range of f is $(0,2]$.

Q1aii Equation of f: $y=2e^{-t}$, equation of f^{-1}: $t=2e^{-y}$.

Re-express with y as the subject: $e^{-y}=\frac{t}{2}$, $e^{y}=\frac{2}{t}$,

$\therefore y=\log_e\left(\frac{2}{t}\right)$.

Q1bi $g(t)=(t-1)^2e^{-t}$, $t \geq 0$.

$g'(t)=2(t-1)e^{-t}-(t-1)^2e^{-t}=\left(-t^2+4t-3\right)e^{-t}$.

$\therefore b=4$, $c=-3$.

Q1bii At stationary points, $g'(t)=0$, $\therefore \left(-t^2+4t-3\right)e^{-t}=0$.

Since $e^{-t}\neq 0$, $\therefore \left(-t^2+4t-3\right)=0$, $(-t+1)(t-3)=0$,

$\therefore t=1$ and $y=0$ or $t=3$ and $y=4e^{-3}$.

Hence $p=0$, $m=3$ and $n=4e^{-3}$.

Q1biii $q(t)$ is a transformation of $g(t)$: Vertical dilation by a factor of 2 and then downward translation by 5 units. These transformations affect only the y-coordinates of points.

$(1,0)\to(1,(2\times 0-5))$, i.e. $(1,-5)$.

$\left(3,4e^{-3}\right)\to\left(3,\left(2\times 4e^{-3}-5\right)\right)$, i.e. $\left(3,\left(8e^{-3}-5\right)\right)$.

Q1ci $h(t)=\left(t^2+at+10\right)e^{-t}$;

$h'(t)=\left(-t^2+(2-a)t+(a-10)\right)e^{-t}$.

At stationary points, $h'(t)=0$.

$\therefore \left(-t^2+(2-a)t+(a-10)\right)e^{-t}=0$.

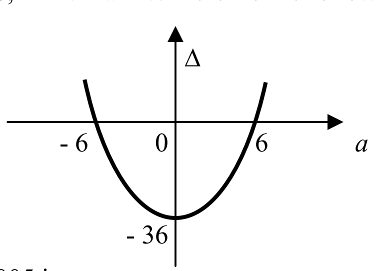
$\therefore -t^2+(2-a)t+(a-10)=0$. To have exactly one solution to this equation, $\Delta=(2-a)^2+4(a-10)=0$, i.e. $a^2-36=0$.

Hence $a=\pm 6$.

Q1cii Since $e^{-t}>0$,

for $h'(t)<0$, $-t^2+(2-a)t+(a-10)<0$.

$\therefore -t^2+(2-a)t+(a-10)$ does not cross the t-axis. Hence its $\Delta=a^2-36<0$, $\therefore -6<a<6$. Refer to the following graph.



Q2a $\Pr(X>81.80)=0.41207\approx 0.412$

$\Pr(81.80<X<90.17)=0.393409\approx 0.393$

$\Pr(X>90.17)=0.018661\approx 0.019$.

Q2b $\Pr(X\geq M)=0.90$, $\therefore \Pr(X<M)=0.10$,

$\therefore M=75.03302\approx 75.03$

Q2c Conditional probability:

$\Pr(X\geq 81.80 \mid X<90.17)=\frac{\Pr(81.80\leq X<90.17)}{\Pr(X<90.17)}$

$=\frac{0.393409}{1-0.018661}=0.40089\approx 0.401$.

Q2d For each throw,

$E(reward)=0\times 0.5+1000\times(1-0.41207-0.5)+2000\times 0.393409$
$+10000\times 0.018661=1061.36\approx \$1060$.

Q2ei For 5 throws,

$E(totalreward)=5\times 1061.36=5306.79\approx \$5310$.

Q2eii Binomial distribution: $n=5$, $p=0.41207$,

$\Pr(X\geq 3)=1-\Pr(X\leq 2)=1-0.661501=0.338499\approx 0.338$.

Q2eiii $E(X)=np=5\times 0.41207=2.06035\approx 2.06$.

Q2eiv There must be at least one over the Olympic Record or 5 between A Standard and Olympic Record in order to earn at least \$10000.

Binomial: $n=5$, $p=0.018661$,

$\Pr(X\geq 1)=1-\Pr(X=0)=1-0.9101=0.08989$

Binomial: $n=5$, $p=0.393409$,

$\Pr(X=5)=0.0094237$

Required probability $=0.08989+0.0094237=0.0993\approx 0.099$.

Q3a Vertical distance between the mountain top and the tunnel = max value of $y=100+50=150$ metres.

Q3b Vertical distance between the valley bottom and the bridge = $|\text{min value of } y|=|-100+50|=50$ metres.

Q3ci When $y=0$, $100\cos\left[\frac{\pi(x-400)}{600}\right]+50=0$,

$\cos\left[\frac{\pi(x-400)}{600}\right]=-\frac{50}{100}=-\frac{1}{2}$,

$\frac{\pi(x-400)}{600}=\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$. $\therefore x=800$.

Length of tunnel = 800 metres.

Q3cii Length of bridge $=2\left(600-\frac{1}{2}\times 800\right)=400$ metres.

Q3d When $y=20,\ 100\cos\left[\frac{\pi(x-400)}{600}\right]+50=20$,

$\cos\left[\frac{\pi(x-400)}{600}\right]=-\frac{30}{100}=-0.3$,

$\frac{\pi(x-400)}{600}=\cos^{-1}(-0.3)=-1.8755$ or 1.8755.

$\therefore x=41.806$ or 758.192.

Length of tunnel $=758.192-41.806=716.386\approx 716$ metres.

Q3e Start of the dam wall $x=800$, middle of the dam wall $x=1000$. $Area=-2\int 100\cos\left[\frac{\pi(x-400)}{600}\right]+50dx$

$$=-2\left[\frac{100\sin\left[\frac{\pi(x-400)}{600}\right]}{\frac{\pi}{600}}+50x\right]_{800}^{1000}$$

$$=-2\left(\frac{60000\sin(\pi)}{\pi}+50000-\frac{60000\sin\left(\frac{2\pi}{3}\right)}{\pi}-40000\right)$$

$=13080\text{ m}^2$.

Q3fi Length of tunnel $=800-2k=2(400-k)$ metres.

Q3fii

Length of bridge $=(600-(400-k))\times 2=2(200+k)$ metres.

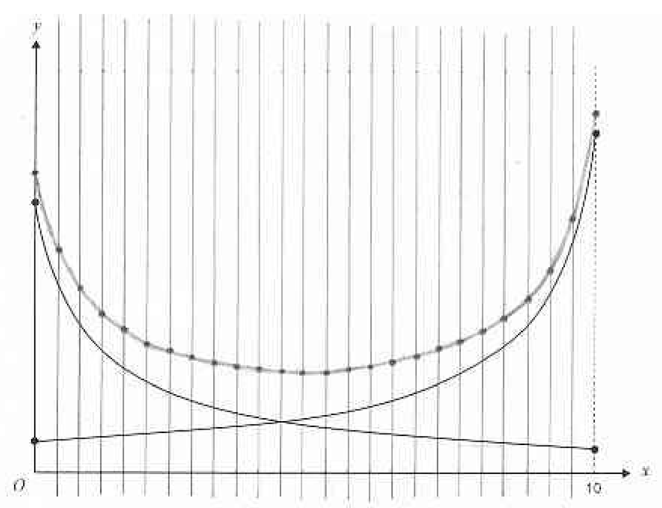
Q3fiii

$C=(2(400-k))^2+(2(200+k))^2=4(400-k)^2+4(200+k)^2$ thousand dollars.

Q3fiv $\frac{dC}{dk}=-8(400-k)+8(200+k)=16(k-100)$. Cost is minimum when $16(k-100)=0$, i.e. $k=100$.

Q4a At $x=3,\ y=\frac{p}{4}+\frac{q}{8}=\frac{2p+q}{8}$.

Q4b Addition of ordinates.



Q4c $y=9(x+1)^{-1}+4(11-x)^{-1}$,

$\frac{dy}{dx}=-9(x+1)^{-2}+4(11-x)^{-2}=-\frac{9}{(x+1)^2}+\frac{4}{(11-x)^2}$.

At the minimum, $-\frac{9}{(x+1)^2}+\frac{4}{(11-x)^2}=0$,

$4(x+1)^2-9(11-x)^2=0$,

$\therefore[2(x+1)-3(11-x)][2(x+1)+3(11-x)]=0$,

or $(5x-31)(35-x)=0$, where $0\le x\le 10$.

Q4di $(5x-31)(35-x)=0$, where $0\le x\le 10$.

$\therefore x=\frac{31}{5}=6.200$ and $y=\frac{9}{6.2+1}+\frac{4}{11-6.2}=2.083$.

Q4dii $\frac{9}{x+1}+\frac{4}{11-x}<5$, and $0\le x\le 10$.

Use graphics calculator to sketch $y=\frac{9}{x+1}+\frac{4}{11-x}$ and $y=5$.

Find the x-coordinates of the intersections. The second one is outside the domain. $\therefore 0.95577<x\le 10$.

Hence length of journey $=10-0.95577=9.04423\approx 9.044$ km.

Q4e $\int_0^{10}\left(\frac{9}{x+1}+\frac{4}{11-x}\right)dx=[9\log_e(x+1)-4\log_e(11-x)]_0^{10}$

$=[9\log_e 11-4\log_e 1-9\log_e 1+4\log_e 11]=13\log_e 11=31.17$.

Total pollution is 31.17.

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