

Q1a $\int \frac{dx}{49+x^2} = \frac{1}{7} \int \frac{7}{7^2+x^2} dx = \frac{1}{7} \tan^{-1}\left(\frac{x}{7}\right) + c.$

Q1b $u = x^4 + 8, \frac{du}{dx} = 4x^3.$

$\int x^3 \sqrt{x^4 + 8} dx = \int \frac{1}{4} \sqrt{u} \frac{du}{dx} dx = \int \frac{1}{4} u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(x^4 + 8)^{\frac{3}{2}}}{6} + c$

Q1c $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \frac{5}{3} \times \frac{\sin 5x}{5x} = \frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = \frac{5}{3}.$

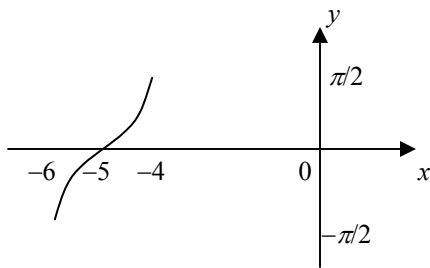
Q1d $\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} - 1 = \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{(\sin \theta + \cos \theta)} - 1$
 $= (\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta) - 1 = (1 - \sin \theta \cos \theta) - 1$
 $= -\sin \theta \cos \theta = -\frac{1}{2} \sin 2\theta.$

Q1e $y = x^3, \frac{dy}{dx} = 3x^2 = 12, \therefore x = \pm 2$ and $y = \pm 8.$
 $y = 12x + b, 8 = 12(2) + b, b = -16.$
 $-8 = 12(-2) + b, b = 16.$

Q2ai $f(x) = \sin^{-1}(x+5)$ is defined for $-1 \leq x+5 \leq 1,$
 i.e. $-6 \leq x \leq -4.$ Domain: $[-6, -4],$ range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

Q2aii $f'(x) = \frac{1}{\sqrt{1-(x+5)^2}}, f'(-5) = \frac{1}{\sqrt{1-(-5+5)^2}} = 1.$

Q2aiii



Q2bi $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n,$

$\frac{d}{dx}(1+x)^n = \frac{d}{dx} \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n \right]$

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \dots + r\binom{n}{r}x^{r-1} + \dots + n\binom{n}{n}x^{n-1}.$

Q2bii Let $x = 2,$

$n3^{n-1} = \binom{n}{1} + \dots + r\binom{n}{r}2^{r-1} + \dots + n\binom{n}{n}2^{n-1} = \sum_{r=1}^n r\binom{n}{r}2^{r-1}.$

Q2ci Given $y = \frac{1}{2}(p+r)x - apr.$ At point $U, x = 0,$
 $\therefore y = -apr,$ hence $U(0, -apr).$

Q2cii Given the tangent at $P(2ap, ap^2)$ is $y = px - ap^2,$ by symmetry the tangent at $Q(2aq, aq^2)$ is $y = qx - aq^2.$

At point $T, px - ap^2 = qx - aq^2, \therefore px - qx = ap^2 - aq^2,$
 $(p-q)x = a(p-q)(p+q), \therefore x = a(p+q),$ where $p \neq q.$
 $\therefore y = px - ap^2 = pa(p+q) - ap^2 = apq.$

Hence $T(a(p+q), apq).$

Q2ciii Since $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ are two different points having the same y -coordinate, $\therefore q^2 = r^2$ and $q \neq r,$
 $\therefore q = -r.$

$\therefore U(0, -apr)$ and $T(a(p+q), apq)$ are two different points having the same y -coordinate, $\therefore TU$ is a horizontal line perpendicular to the vertical axis of the parabola.

Q3a $\int_0^{\frac{\pi}{4}} \sin^2 x dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2x) dx = \left[\frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) \right]_0^{\frac{\pi}{4}}$
 $= \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi - 2}{8}.$

Q3bi $f(x) = 3 \log_e x - x, f(1.5) = 3 \log_e 1.5 - 1.5 = -0.2836,$
 $f(2) = 3 \log_e 2 - 2 = 0.0794. \therefore f(x)$ has a zero at $1.5 < x < 2,$
 $\therefore x$ -coordinate of P is $1.5 < x < 2.$

Q3bii $f'(x) = \frac{3}{x} - 1, x_1 = 1.5, f'(1.5) = \frac{3}{1.5} - 1 = 1.$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{-0.2836}{1} = 1.78.$

Q3ci Three blocks high: $5 \times 4 \times 3 = 60$

Q3cii Total number of 2, 3, 4 and 5-block towers:
 $5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1 = 320$

Q3di Since $\angle QKT + \angle QMT = 180^\circ, \therefore QKTM$ is a cyclic quadrilateral.

Q3dii $\angle KMT$ and $\angle KQT$ are subtended by the same arc $KT,$
 \therefore they are equal.

Q3diii $\angle PTN = \angle PQT$ (The angle between a chord and a tangent at an end of the chord equals the angle in the alternate segment).

$\angle PTN = \angle PQT = \angle KQT = \angle KMT,$ i.e. $\angle PTN$ and $\angle KMT$ are equal corresponding angles, $\therefore MK \parallel TP.$

Q4ai and Q4aii $P(x) = x^3 + rx^2 + sx + t = (x-1)(x-\alpha)(x+\alpha)$
 $= x^3 - x^2 - \alpha^2x + \alpha^2$.

Comparing coefficients: $r = -1$, $s = -\alpha^2$, $t = \alpha^2$.
 $\therefore s + t = 0$.

Q4bi $T = \frac{2\pi}{n}$, $5 = \frac{2\pi}{n}$, $\therefore n = \frac{2\pi}{5}$. $\therefore x = 18 \cos\left(\frac{2\pi}{5}t\right)$.

Q4bii A rest position is at one of the extreme positions.

First time at $x = 9$, $9 = 18 \cos\left(\frac{2\pi}{5}t\right)$, $\therefore \cos\left(\frac{2\pi}{5}t\right) = 0.5$,

$\frac{2\pi}{5}t = \frac{\pi}{3}$, $t = \frac{5}{6}$ s. Required time is $\frac{5}{6}$ s.

Q4ci $\ddot{x} = 18x^3 + 27x^2 + 9x$. At $t = 0$, $x = -2$, $v = -6$.

$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 18x^3 + 27x^2 + 9x$,

$\frac{1}{2}v^2 = \int (18x^3 + 27x^2 + 9x) dx$, $\frac{1}{2}v^2 = \frac{9x^4}{2} + 9x^3 + \frac{9x^2}{2} + d$,

$\therefore v^2 = 9x^4 + 18x^3 + 9x^2 + c$.

Using $x = -2$, $v = -6$, c is 0.

$\therefore v^2 = 9x^4 + 18x^3 + 9x^2 = 9x^2(x^2 + 2x + 1) = 9x^2(x+1)^2$.

Q4cii At $t = 0$, $x = -2$, $v = -6$, $\ddot{x} = -54$, $\therefore v$ remains negative and $x < -2$ as time progresses.

$\therefore v = -\sqrt{9x^2(x+1)^2} = -3x(1+x)$, i.e. $\frac{dx}{dt} = -3x(1+x)$.

Hence $\frac{dt}{dx} = -\frac{1}{3} \times \frac{1}{x(1+x)}$, $t = -\frac{1}{3} \int \frac{1}{x(1+x)} dx$,

$\therefore \int \frac{1}{x(1+x)} dx = -3t$.

Q4ciii $\log_e\left(1 + \frac{1}{x}\right) = 3t + c$. At $t = 0$, $x = -2$,

$\therefore \log_e\left(1 - \frac{1}{2}\right) = c$, $c = \log_e \frac{1}{2} = -\log_e 2$.

$\therefore \log_e\left(1 + \frac{1}{x}\right) = 3t - \log_e 2$, $\log_e\left(1 + \frac{1}{x}\right) + \log_e 2 = 3t$,

$\therefore \log_e 2\left(1 + \frac{1}{x}\right) = 3t$, $2\left(1 + \frac{1}{x}\right) = e^{3t}$, $1 + \frac{1}{x} = 0.5e^{3t}$,

$\frac{1}{x} = 0.5e^{3t} - 1$, $x = \frac{1}{0.5e^{3t} - 1}$.

Q5a $y = 10e^{-0.7t} + 3$, $\frac{dy}{dt} = -0.7(10e^{-0.7t}) = -0.7(y-3)$.

Q5b All functions (relations) have an inverse. Does the question mean to show that $f(x) = \log_e(1 + e^x)$ has an inverse **function**?

$f'(x) = \frac{e^x}{1+e^x} > 0$ for all x , $\therefore f(x)$ is an increasing function for

all x , $\therefore f(x)$ is a one-to-one function and hence $f^{-1}(x)$ exists.

Q5ci $V = \frac{\pi}{3}x^2(3r-x) = \pi rx^2 - \frac{\pi}{3}x^3$, $\frac{dV}{dx} = 2\pi rx - \pi x^2$.

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$, $\therefore \frac{dx}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dx}} = \frac{k}{2\pi rx - \pi x^2} = \frac{k}{\pi x(2r-x)}$.

Q5cii Since the rate is constant, $V = kt$, $t = \frac{V}{k}$.

When $x = \frac{1}{3}r$, $t_i = \frac{8\pi}{81k}r^3$.

When $x = \frac{2}{3}r$, $t_f = \frac{28\pi}{81k}r^3$.

$\therefore \frac{t_f}{t_i} = \frac{28}{8} = 3.5$

Q5di $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$,

$\therefore 1 + \tan \alpha \tan \beta = \frac{\tan \alpha - \tan \beta}{\tan(\alpha - \beta)}$.

Let $\alpha = (n+1)\theta$ and $\beta = n\theta$.

$1 + \tan(n+1)\theta \tan n\theta = \frac{\tan(n+1)\theta - \tan n\theta}{\tan \theta}$,

$\therefore 1 + \tan(n+1)\theta \tan n\theta = \cot \theta (\tan(n+1)\theta - \tan n\theta)$,

$\therefore 1 + \tan n\theta \tan(n+1)\theta = \cot \theta (\tan(n+1)\theta - \tan n\theta)$.

Q5dii Given $1 + \tan n\theta \tan(n+1)\theta = \cot \theta (\tan(n+1)\theta - \tan n\theta)$,

prove by induction that for $n \geq 1$,

$\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \dots + \tan n\theta \tan(n+1)\theta$

$= -(n+1) + \cot \theta \tan(n+1)\theta$

It is true for $n = 1$: $1 + \tan \theta \tan 2\theta = \cot \theta (\tan 2\theta - \tan \theta)$,

$1 + \tan \theta \tan 2\theta = \cot \theta \tan 2\theta - \cot \theta \tan \theta$,

$\therefore \tan \theta \tan 2\theta = -2 + \cot \theta \tan 2\theta$.

Assume that it is true for $n = k$, it is also true for $n = k + 1$:

$\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \dots + \tan k\theta \tan(k+1)\theta$

$= -(k+1) + \cot \theta \tan(k+1)\theta$

then $\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \dots + \tan k\theta \tan(k+1)\theta + \tan(k+1)\theta \tan(k+2)\theta$

$= -(k+1) + \cot \theta \tan(k+1)\theta + \tan(k+1)\theta \tan(k+2)\theta$

$= -(k+1) + \cot \theta \tan(k+1)\theta + \cot \theta (\tan(k+2)\theta - \tan(k+1)\theta) - 1$

$= -((k+1)+1) + \cot \theta \tan((k+1)+1)\theta$.

\therefore it is true for all $n \geq 1$.

Q6ai $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (a - Vt \cos \theta)^2 + (Vt - Vt \sin \theta)^2$

$= a^2 - 2aVt \cos \theta + V^2t^2 \cos^2 \theta + V^2t^2 - 2V^2t^2 \sin \theta + V^2t^2 \sin^2 \theta$

$= V^2t^2 \sin^2 \theta + V^2t^2 \cos^2 \theta + V^2t^2 - 2V^2t^2 \sin \theta - 2aVt \cos \theta + a^2$

$= V^2t^2 (\sin^2 \theta + \cos^2 \theta) + V^2t^2 - 2V^2t^2 \sin \theta - 2aVt \cos \theta + a^2$

$= 2V^2t^2(1 - \sin \theta) - 2aVt \cos \theta + a^2$

Q6aii Smallest distance occurs at where $\frac{d(L^2)}{dt} = 0$,

$$4V^2(1 - \sin \theta)t - 2aV \cos \theta = 0, \therefore t = \frac{a \cos \theta}{2V(1 - \sin \theta)}.$$

$$L = \sqrt{2V^2t^2(1 - \sin \theta) - 2aVt \cos \theta + a^2},$$

$$L_{\min} = \sqrt{2V^2 \left(\frac{a \cos \theta}{2V(1 - \sin \theta)} \right)^2 (1 - \sin \theta) - 2aV \frac{a \cos \theta}{2V(1 - \sin \theta)} \cos \theta + a^2}$$

$$= \sqrt{\frac{a^2 \cos^2 \theta}{2(1 - \sin \theta)} - \frac{a^2 \cos^2 \theta}{(1 - \sin \theta)} + a^2}$$

$$= a \sqrt{1 - \frac{\cos^2 \theta}{2(1 - \sin \theta)}} = a \sqrt{1 - \frac{1 - \sin^2 \theta}{2(1 - \sin \theta)}}$$

$$= a \sqrt{1 - \frac{(1 - \sin \theta)(1 + \sin \theta)}{2(1 - \sin \theta)}} = a \sqrt{1 - \frac{1 + \sin \theta}{2}} = a \sqrt{\frac{1 - \sin \theta}{2}}.$$

Q6aiii Particle 1 is ascending when its $\frac{dy}{dt} > 0$,

i.e. $V \sin \theta - gt > 0$, $t < \frac{V \sin \theta}{g}$.

$$\therefore \frac{a \cos \theta}{2V(1 - \sin \theta)} < \frac{V \sin \theta}{g}, \therefore V^2 > \frac{ag \cos \theta}{2 \sin \theta (1 - \sin \theta)},$$

$$\therefore V > \sqrt{\frac{ag \cos \theta}{2 \sin \theta (1 - \sin \theta)}}.$$

Q6bi Four-member team: Let Y be the number of competitors not completing the course.

$$\Pr(Y \geq 3) = \Pr(Y = 3) + \Pr(Y = 4) = \binom{4}{3} p q^3 + q^4 = 4 p q^3 + q^4.$$

Q6bii Four-member team. Let X be the number of competitors completing the course.

$$\Pr(X \geq 2) = \binom{4}{2} p^2 q^2 + \binom{4}{3} p^3 q + p^4$$

$$= 6(1 - q)^2 q^2 + 4(1 - q)^3 q + (1 - q)^4$$

$$= (1 - q)^2 (6q^2 + 4q(1 - q) + (1 - q)^2)$$

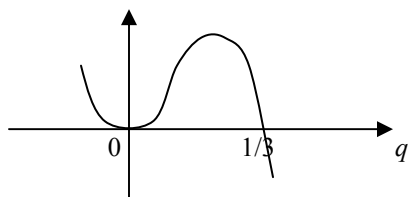
$$= (1 - q)^2 (1 + 2q + 3q^2).$$

Q6biii Two-member team: $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - q^2$.

Q6biv $1 - q^2 > (1 - q)^2 (1 + 2q + 3q^2)$,

$$(1 - q)(1 + q) > (1 - q)^2 (1 + 2q + 3q^2),$$

$$1 + q > (1 - q)(1 + 2q + 3q^2), \therefore q^2 - 3q^3 < 0, q^2(1 - 3q) < 0.$$



$$q^2(1 - 3q) < 0 \text{ when } q > \frac{1}{3}, \therefore \frac{1}{3} < q < 1.$$

Q7a $A = \frac{1}{2} r^2 (2\theta) - \frac{1}{2} r^2 \sin 2\theta = r^2 \theta - \frac{1}{2} r^2 (2 \sin \theta \cos \theta)$
 $= r^2 (\theta - \sin \theta \cos \theta).$

Q7b $w = r(2\theta)$, $r = \frac{w}{2\theta}$, $\therefore A = \frac{w^2}{4\theta^2} (\theta - \sin \theta \cos \theta).$

$$\frac{dA}{d\theta} = -\frac{2w^2}{4\theta^3} (\theta - \sin \theta \cos \theta) + \frac{w^2}{4\theta^2} (1 - \cos^2 \theta + \sin^2 \theta)$$

$$= -\frac{2w^2}{4\theta^3} (\theta - \sin \theta \cos \theta) + \frac{w^2}{4\theta^2} (1 - \cos^2 \theta + 1 - \cos^2 \theta)$$

$$= -\frac{2w^2}{4\theta^3} (\theta - \sin \theta \cos \theta) + \frac{2w^2}{4\theta^2} (1 - \cos^2 \theta)$$

$$= \frac{w^2}{2\theta^3} (\sin \theta \cos \theta - \theta \cos^2 \theta)$$

$$= \frac{w^2 \cos \theta (\sin \theta - \theta \cos \theta)}{2\theta^3}.$$

Q7c Let $g(\theta) = \sin \theta - \theta \cos \theta$,

$$g'(\theta) = \cos \theta - \cos \theta + \theta \sin \theta = \theta \sin \theta.$$

For $0 < \theta < \pi$, $g'(\theta) = \theta \sin \theta > 0$,

$\therefore g(\theta) = \sin \theta - \theta \cos \theta$ is an increasing function in $0 < \theta < \pi$,

and $g(0) = 0$, hence $g(\theta) = \sin \theta - \theta \cos \theta > 0$ for $0 < \theta < \pi$.

Q7d $\frac{dA}{d\theta} = 0$, $\frac{w^2 \cos \theta (\sin \theta - \theta \cos \theta)}{2\theta^3} = 0$, where $0 < \theta < \pi$.

Since $w > 0$ and $\sin \theta - \theta \cos \theta > 0$,

$\therefore \cos \theta = 0$ and $\theta = \frac{\pi}{2}$ is the only solution in $0 < \theta < \pi$.

Q7e

θ	1	$\frac{\pi}{2}$	2
$\frac{dA}{d\theta}$	+	0	-

$\therefore A$ is maximum when $\theta = \frac{\pi}{2}$.

$$\therefore A = \frac{w^2}{4\theta^2} (\theta - \sin \theta \cos \theta),$$

$$A_{\max} = \frac{w^2}{4\left(\frac{\pi}{2}\right)^2} \left(\frac{\pi}{2} - \sin \frac{\pi}{2} \cos \frac{\pi}{2} \right) = \frac{w^2}{2\pi} \text{ square units.}$$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.