

Q1a $(1 + \sqrt{5})^3 = 1^3 + 3(1)^2(\sqrt{5}) + 3(1)(\sqrt{5})^2 + (\sqrt{5})^3 = 16 + 8\sqrt{5}$.

Q1b $x = \frac{(4)(3) + (19)(2)}{2+3} = 10$; $y = \frac{(5)(3) + (-5)(2)}{2+3} = 1$.

Q1c $\frac{d}{dx}(\tan^{-1}(x^4)) = \left(\frac{d}{d(x^4)}(\tan^{-1}(x^4))\right)\left(\frac{d}{dx}(x^4)\right)$
 $= \left(\frac{1}{1+(x^4)^2}\right)(4x^3) = \frac{4x^3}{1+x^8}$.

Q1d Line 1: $x - 2y + 3 = 0$, $y = \frac{1}{2}x + \frac{3}{2}$, $\frac{dy}{dx} = \frac{1}{2}$, $m_1 = \frac{1}{2}$.

Curve 2: $y = x^3 + 1$, $\frac{dy}{dx} = 3x^2$. At $x = 1$, $\frac{dy}{dx} = 3$, $m_2 = 3$.

$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{3 - \frac{1}{2}}{1 + \frac{3}{2}} = 1$, $\therefore \theta = \frac{\pi}{4}$.

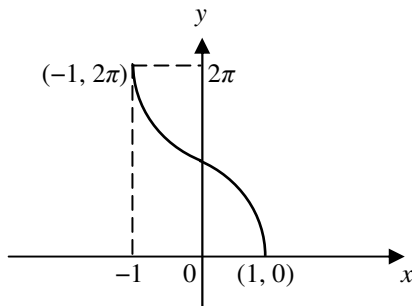
Q1e Given $u = 25 - x^2$, $\therefore -\frac{du}{dx} = 2x$.

When $x = 3$, $u = 16$; when $x = 4$, $u = 9$.

$\int_3^4 \frac{2x}{\sqrt{25-x^2}} dx = -\int_3^4 \frac{1}{\sqrt{u}} \frac{du}{dx} dx = -\int_{16}^9 u^{-\frac{1}{2}} du = \int_9^{16} \frac{1}{2} u^{-\frac{1}{2}} du$
 $= \left[2u^{\frac{1}{2}}\right]_9^{16} = 2$.

Q2a $LHS = \frac{1 - \cos \theta}{\sin \theta} = \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2} = RHS$.

Q2bi



Q2bii The range is $[0, 2\pi]$.

Q2c $P(x) = x^2 + ax + b$,

$P(2) = 4 + 2a + b = 0$, $\therefore 2a + b = -4$.

$P(-1) = 1 - a + b = 18$, $\therefore a - b = -17$.

Solve simultaneously, $a = -7$ and $b = 10$.

Q2di $v = 50(1 - e^{-0.2t})$, $a = \frac{dv}{dt} = 50(0.2e^{-0.2t}) = 10e^{-0.2t}$.

When $t = 10$, $a = 10e^{-2} \approx 1.4 \text{ ms}^{-2}$.

Q2dii Displacement in the first 10 seconds

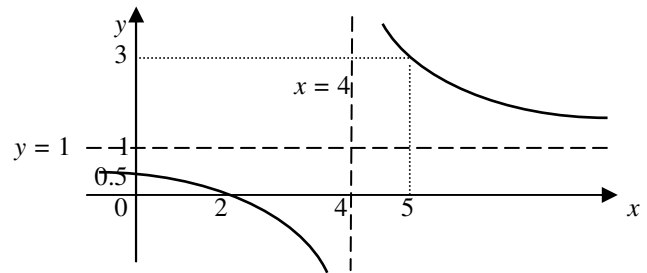
$= \int_0^{10} 50(1 - e^{-0.2t}) dt = [50(t + 5e^{-0.2t})]_0^{10} = 50(10 + 5e^{-2} - 5) \approx 284$.

Distance fallen = 284 m.

Q3a $V = \int_0^3 \pi y^2 dx = \int_0^3 \frac{\pi}{9+x^2} dx = \left[\frac{\pi}{3} \tan^{-1} \frac{x}{3}\right]_0^3 = \frac{\pi}{3} \tan^{-1} 1 = \frac{\pi^2}{12}$

Q3bi $y = \frac{x-2}{x-4} = 1 - \frac{2}{x-4}$. Vertical asymptote: $x = 4$;

horizontal asymptote: $y = 1$.



Q3bii $\frac{x-2}{x-4} = 3$ when $x-2 = 3x-12$, i.e. $x = 5$.

From graph, $\frac{x-2}{x-4} \leq 3$ when $x < 4$ or $x \geq 5$.

Q3ci $\ddot{x} = -e^{-2x}$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-2x}$,

$\frac{1}{2}v^2 = \int(-e^{-2x}) dx = \frac{1}{2}e^{-2x} + c_1$.

Given $x = 0$ and $v = 1$ when $t = 0$,

$\therefore c_1 = 0$, $\therefore v^2 = e^{-2x}$, $\therefore \dot{x} = v = e^{-x}$. Note: $\dot{x} = -e^{-x}$ does not meet the initial conditions.

Q3cii $\dot{x} = e^{-x}$, $\therefore \frac{dx}{dt} = e^{-x}$, $\frac{dt}{dx} = e^x$, $\therefore t = \int e^x dx = e^x + c_2$.

Given $x = 0$ and $v = 1$ when $t = 0$,

$\therefore c_2 = -1$ and $t = e^x - 1$. Hence $x = \log_e(t+1)$.

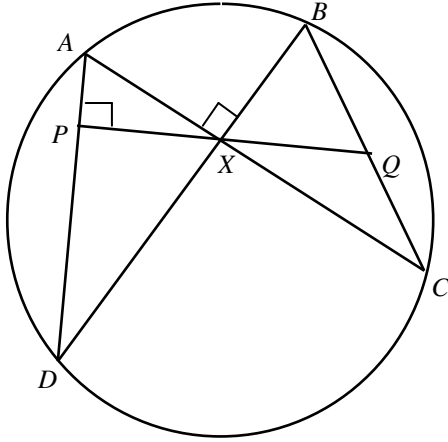
Q4ai $\Pr(X = 2) = 0.1^2 = 0.01$

Q4aaii $\Pr(X = 2) = {}^{20}C_2(0.1)^2(0.9)^{18} \approx 0.285$

Q4aiii $\Pr(X > 2) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$
 $= 1 - 0.9^{20} - {}^{20}C_1(0.1)(0.9)^{19} - 0.285 \approx 0.32$.

Q4b For $n=1$, $7^{2n-1} + 5 = 12$ is divisible by 12.
 For $n=k$, assume that $7^{2k-1} + 5$ is divisible by 12.
 For $n=k+1$,
 $7^{2(k+1)-1} + 5 = 7^2 7^{2k-1} + 5 = 49(7^{2k-1} + 5) - 48 \times 5$ is divisible by 12 because both $7^{2k-1} + 5$ and 48 are divisible by 12.
 Hence $7^{2n-1} + 5 = 12$ is divisible by 12 for all integers $n \geq 1$.

Q4ci



$\angle QBX = \angle XAP$ (angles subtended on the circumference by the same arc).
 $\angle XAP + \angle AXP = \angle PXD + \angle AXP = 90^\circ$, $\therefore \angle XAP = \angle PXD$.
 $\angle PXD = \angle QXB$ (vertically opposite angles).
 $\therefore \angle QXB = \angle QBX$.

Q4cii $\angle QXB = \angle QBX$, $\therefore \Delta QXB$ is isosceles and $QB = QX$.
 Similarly, $\angle QXC = \angle QCX$, $\therefore \Delta QXC$ is isosceles and $QX = QC$. $\therefore QB = QC$ and hence Q bisects BC .

Q5ai Area of sector $OPQ = \frac{1}{2} r^2 \theta$.

Area of $\Delta OPT = \frac{1}{2} r(PT) = \frac{1}{2} r(r \tan \theta) = \frac{1}{2} r^2 \tan \theta$.

$\therefore \frac{1}{2} r^2 \tan \theta = 2 \times \frac{1}{2} r^2 \theta$, $\therefore \tan \theta = 2\theta$.

Q5aai Let $f(\theta) = 2\theta - \tan \theta$. $f'(\theta) = 2 - \sec^2 \theta$.

Newton's method:

$$\theta_1 \approx \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)} = 1.15 - \frac{2 \times 1.15 - \tan 1.15}{2 - \sec^2 1.15} = 1.1664.$$

Q5b There are $4!$ permutations of the four children together, and there are $3!$ ways in arranging a particular permutation of the four children, Mr Roberts and Mrs Roberts.

\therefore total number of arrangements of the family of six with the children together is $3!4!$.

Total number of arrangements of the family of six without restriction is $6!$.

$$\therefore \Pr(\text{children together}) = \frac{3!4!}{6!} = \frac{1}{5}.$$

$$\text{Q5c } \sin^{-1} x + \frac{1}{2} \cos^{-1} y = \frac{\pi}{3} \quad (1)$$

$$3 \sin^{-1} x - \frac{1}{2} \cos^{-1} y = \frac{2\pi}{3} \quad (2)$$

$$(1) + (2): 4 \sin^{-1} x = \pi, \sin^{-1} x = \frac{\pi}{4}, \therefore x = \frac{1}{\sqrt{2}}.$$

$$\text{From (1), } \cos^{-1} y = 2 \left(\frac{\pi}{3} - \sin^{-1} x \right) = 2 \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\pi}{6},$$

$$\therefore y = \frac{\sqrt{3}}{2}.$$

Q5di Substitute $x = 2aq$ and $y = aq^2$ into the equation of PQ :

$$2aq + paq^2 - 2ap - ap^3 = 0,$$

$$a[2(q-p) + p(q^2 - p^2)] = 0,$$

$$a[2(q-p) + p(q-p)(q+p)] = 0,$$

$$a(q-p)[2 + p(q+p)] = 0.$$

Since $q-p \neq 0 \therefore q \neq p$, and $a \neq 0$,

$$\therefore 2 + p(q+p) = 0, \text{ hence } p^2 + pq + 2 = 0.$$

$$\text{Q5dii Gradient of } OP = \frac{ap^2}{2ap} = \frac{p}{2},$$

$$\text{gradient of } OQ = \frac{aq^2}{2aq} = \frac{q}{2}.$$

If $OP \perp OQ$, then $\frac{p}{2} \times \frac{q}{2} = -1$, $\therefore pq = -4$.

From Q5di, $p^2 + pq + 2 = 0$, $\therefore p^2 - 4 + 2 = 0$, hence $p^2 = 2$.

$$\text{Q6ai } x = \sqrt{3} \sin 2t - \cos 2t + 3, \dot{x} = 2\sqrt{3} \cos 2t + 2 \sin 2t,$$

$$\ddot{x} = -4\sqrt{3} \sin 2t + 4 \cos 2t = -4(\sqrt{3} \sin 2t - \cos 2t) = -4(x-3).$$

$$\text{Q6aai Period} = \frac{2\pi}{2} = \pi \text{ seconds.}$$

$$\text{Q6aiii } \dot{x} = 2\sqrt{3} \cos 2t + 2 \sin 2t,$$

$$\dot{x} = A \cos(2t - \alpha) = A \cos 2t \cos \alpha + A \sin 2t \sin \alpha.$$

$$\therefore A \cos \alpha = 2\sqrt{3} \text{ and } A \sin \alpha = 2.$$

$$\text{Hence } A^2 \cos^2 \alpha + A^2 \sin^2 \alpha = 16,$$

$$\therefore A = 4 \text{ and } \sin \alpha = \frac{1}{2}, \text{ i.e. } \alpha = \frac{\pi}{6}.$$

$$\therefore \dot{x} = 4 \cos \left(2t - \frac{\pi}{6} \right).$$

$$\text{Q6aiv } 0 \leq t \leq \pi, 0 \leq 2t \leq 2\pi.$$

$$\text{When } \dot{x} = 2, 4 \cos \left(2t - \frac{\pi}{6} \right) = 2, \cos \left(2t - \frac{\pi}{6} \right) = \frac{1}{2},$$

$$2t - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3} \therefore t = \frac{\pi}{4}, \frac{11\pi}{12}.$$

$$\text{When } \dot{x} = -2, 4 \cos \left(2t - \frac{\pi}{6} \right) = -2, \cos \left(2t - \frac{\pi}{6} \right) = -\frac{1}{2},$$

$$2t - \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3} \therefore t = \frac{5\pi}{12}, \frac{3\pi}{4}.$$

Q6bi $f(x) = e^x - e^{-x}$, $f'(x) = e^x + e^{-x} > 0$ for all values of x ,
 $\therefore f(x)$ is increasing for all values of x .

Q6bii Equation of $f(x)$: $y = e^x - e^{-x}$

Equation of inverse: $x = e^y - e^{-y}$.

$$xe^y = (e^y)^2 - 1, (e^y)^2 - xe^y - 1 = 0.$$

Apply quadratic formula: $e^y = \frac{x + \sqrt{x^2 + 4}}{2}$ since $e^y > 0$.

$$\therefore y = \log_e \left(\frac{x + \sqrt{x^2 + 4}}{2} \right), \therefore f^{-1}(x) = \log_e \left(\frac{x + \sqrt{x^2 + 4}}{2} \right).$$

$$Q6biii \quad y = e^x - e^{-x}, \therefore x = \log_e \left(\frac{y + \sqrt{y^2 + 4}}{2} \right).$$

$$\text{When } y = 5, x = \log_e \left(\frac{5 + \sqrt{5^2 + 4}}{2} \right) = 1.65.$$

$$Q7ai \quad y = kx^n, \frac{dy}{dx} = nkx^{n-1}. \quad y = \log_e x, \frac{dy}{dx} = \frac{1}{x}.$$

$$\text{At } x = a, nka^{n-1} = \frac{1}{a}, \therefore a^n = \frac{1}{nk}.$$

$$Q7aii \quad \text{At } x = a, ka^n = \log_e a.$$

$$\text{From Q7ai, } a^n = \frac{1}{nk}, \therefore k \left(\frac{1}{nk} \right) = \log_e a, \log_e a = \frac{1}{n}, a = e^{\frac{1}{n}}.$$

$$\therefore a^n = e = \frac{1}{nk}. \text{ Hence } k = \frac{1}{en}.$$

$$Q7bi \quad x = 14t \cos \theta, \therefore t = \frac{x}{14 \cos \theta}.$$

$$\begin{aligned} y &= 14t \sin \theta - 4.9t^2, y = 14 \left(\frac{x}{14 \cos \theta} \right) \sin \theta - 4.9 \left(\frac{x}{14 \cos \theta} \right)^2 \\ &= x \tan \theta - \frac{x^2}{40 \cos^2 \theta} = x \tan \theta - \frac{x^2}{40} \sec^2 \theta \\ &= x \tan \theta - \frac{x^2}{40} (1 + \tan^2 \theta) = mx - \left(\frac{1+m^2}{40} \right) x^2, \text{ where } m = \tan \theta. \end{aligned}$$

$$Q7bii \quad \text{At } x = 10 \text{ and when } m = 2 \pm \sqrt{3 - 0.4h},$$

$$\begin{aligned} y &= mx - \left(\frac{1+m^2}{40} \right) x^2 \\ &= 10(2 \pm \sqrt{3 - 0.4h}) - 2.5 \left(1 + (2 \pm \sqrt{3 - 0.4h})^2 \right) = h. \end{aligned}$$

$$\text{Since } 3 - 0.4h \geq 0, h \leq \frac{3}{0.4} = 7.5, \therefore \max h \text{ is } 7.5 \text{ m.}$$

$$Q7biii \quad \text{Given } m = 2 \pm \sqrt{3 - 0.4h}.$$

When $h = 3.9$, $m = 0.8$ or 3.2 .

When $h = 5.9$, $m = 1.2$ or 2.8 .

\therefore The other interval is $0.8 \leq m \leq 1.2$.

Q7biv Let $y = 0$ to find the range.

$$mx - \left(\frac{1+m^2}{40} \right) x^2 = 0, x \left[m - \left(\frac{1+m^2}{40} \right) x \right] = 0,$$

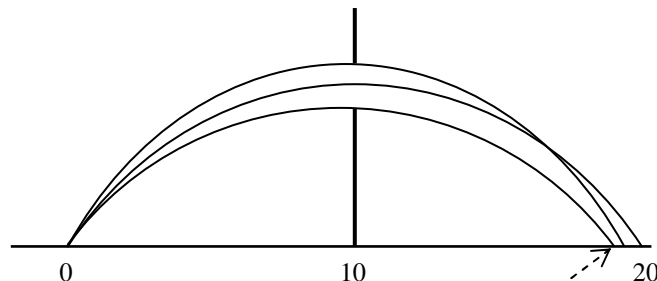
$$\therefore x = 0 \text{ or } x = \frac{40m}{1+m^2}. \therefore \text{the range is } \frac{40m}{1+m^2} \text{ metres.}$$

$$\text{For } 2.8 \leq m \leq 3.2, \frac{40(3.2)}{1+3.2^2} \leq x \leq \frac{40(2.8)}{1+2.8^2},$$

$$11.388 \leq x \leq 12.670.$$

Width of interval = $12.670 - 11.388 \approx 1.282$ metres.

The range is maximum when $\theta^\circ = 45^\circ$, i.e. $m = \tan 45^\circ = 1$, which is within the interval $0.8 \leq m \leq 1.2$.



$$\text{When } m = 0.8, x = \frac{40(0.8)}{1+0.8^2} = 19.512; \text{ when } m = 1, x = 20;$$

$$\text{when } m = 1.2, x = 19.672.$$

$$\text{Width of interval} = 20 - 19.512 \approx 0.488 \text{ metres.}$$

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