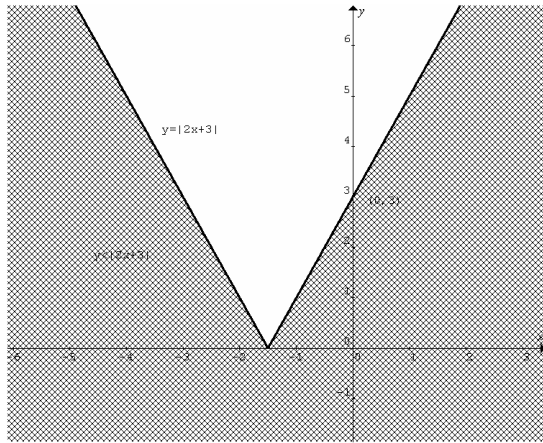


Q1a  $\int \frac{1}{x^2 + 49} dx = \int \frac{1}{7^2 + x^2} dx = \frac{1}{7} \tan^{-1}\left(\frac{x}{7}\right) + C.$

Q1b



Q1c For  $y = \cos^{-1}\left(\frac{x}{4}\right)$ , the domain is  $[-4, 4]$ , the range  $[0, \pi]$ .

Q1d Let  $u = 2x^2 + 1$ ,  $\frac{du}{dx} = 4x$ , or  $x = \frac{1}{4} \frac{du}{dx}$ .  
 $\therefore \int x(2x^2 + 1)^5 dx = \int \frac{1}{4} u^5 \frac{du}{dx} dx = \int \frac{1}{4} u^5 du$   
 $= \frac{1}{9} u^{\frac{9}{4}} + C = \frac{1}{9} (2x^2 + 1)^{\frac{9}{4}} + C.$

Q1e  $\frac{3x^{-1} + 2x}{5} = 1$  and  $\frac{3 \times 8 + 2y}{5} = 4$ ,  $\therefore x = 4$  and  $y = -2$ .

Q1f Let  $\tan \theta = 3$  and  $\tan \phi = m$ ,  $\therefore$  either  $\theta - \phi = 45^\circ$  or  $\phi - \theta = 45^\circ$ . Hence  $\tan(\theta - \phi) = 1$  or  $\tan(\phi - \theta) = 1$ .  
 $\therefore \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \frac{3 - m}{1 + 3m} = 1$  or  $\frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} = \frac{m - 3}{1 + 3m} = 1$ .  
 $\therefore m = \frac{1}{2}$  or  $-2$ .

Q2a  $\frac{d}{dx} (2 \sin^{-1}(5x)) = 5 \times \frac{2}{\sqrt{1 - (5x)^2}} = \frac{10}{\sqrt{1 - 25x^2}}.$

Q2b The term independent of  $x$  in the expansion of  $\left(2x - \frac{1}{x^2}\right)^{12}$  is  ${}^{12}C_4 (2x)^8 \left(-\frac{1}{x^2}\right)^4 = 126720$ .

Q2ci Apply the product rule to find  $\frac{d}{dx} (e^{3x} (\cos x - 3 \sin x))$   
 $= e^{3x} (-\sin x - 3 \cos x) + 3e^{3x} (\cos x - 3 \sin x)$   
 $= -10e^{3x} \sin x.$

Q2cii  $\int -10e^{3x} \sin x dx = e^{3x} (\cos x - 3 \sin x) + D,$   
 $\therefore \int e^{3x} \sin x dx = -\frac{1}{10} e^{3x} (\cos x - 3 \sin x) + C.$

Q2di  $T = 3 + Ae^{-kt}$ ,  $\therefore \frac{dT}{dt} = -kAe^{-kt}$  and  $-k(T - 3) = -kAe^{-kt}$ .  
 $\therefore T = 3 + Ae^{-kt}$  satisfies the differential equation.

Q2dii At  $t = 0$ ,  $T = 25$ . At  $t = 10$ ,  $T = 11$ . Substitute into  $T = 3 + Ae^{-kt}$  to obtain  $25 = 3 + A$ ,  $\therefore A = 22$  and  $11 = 3 + Ae^{-10k}$ , i.e.  $\frac{4}{11} = e^{-10k}$ , or  $k = \frac{1}{10} \log_e \left(\frac{11}{4}\right)$ .  
 At  $t = 15$ ,  $T = 3 + 22e^{-15k} = 7.8^\circ \text{C}.$

Q3ai  $g(0.7) = -0.041$ ,  $g(0.9) = 0.168$ .  $\therefore g(x)$  has a zero between 0.7 and 0.9.

Q3aii Halving the interval,  $x = \frac{0.7 + 0.9}{2} = 0.8$ ,  
 $g(0.8) = 0.052$ ,  $\therefore g(x)$  has a zero between 0.7 and 0.8.  
 Halving the interval,  $x = \frac{0.7 + 0.8}{2} = 0.75$ ,  
 $g(0.75) = 0.003$ ,  $\therefore g(x)$  has a zero between 0.7 and 0.75.  
 To one decimal place,  $\therefore g(x)$  has a zero at 0.7.

Q3bi  $\sin(5x + 4x) + \sin(5x - 4x)$   
 $= \sin 5x \cos 4x + \cos 5x \sin 4x + \sin 5x \cos 4x - \cos 5x \sin 4x$   
 $= 2 \sin 5x \cos 4x.$

Q3bii  $\int \sin 5x \cos 4x dx = \int \frac{1}{2} (\sin 9x + \sin x) dx$   
 $= -\frac{1}{18} \cos 9x - \frac{1}{2} \cos x + C.$

Q3c  $f(x) = x^2 + 5x$ ,  $f(x + h) = (x + h)^2 + 5(x + h)$ ,  
 $f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^2 + 5(x + h) - x^2 - 5x}{h} = \lim_{h \rightarrow 0} (2x + h + 5) = 2x + 5$

Q3di  $EB = 7 - 4 = 3$ ,  $EC = \ell - x$ .  $AE \times EB = DE \times EC$ ,  
 $\therefore 12 = x(\ell - x)$ , or  $x^2 - \ell x + 12 = 0$ .

Q3dii For this quadratic equation to have real  $x$  for its solutions,  $\Delta \geq 0$ , i.e.  $\ell^2 - 4(1)(12) \geq 0$ ,  $\ell^2 \geq 48$ . Since  $\ell > 0$ ,  
 $\therefore \ell \geq \sqrt{48}$ . Hence the shortest chord has length  $\sqrt{48} = 4\sqrt{3}$ .

Q4a Let  $u = \sin x$ , when  $x = 0$ ,  $u = 0$ ; when  $x = \frac{\pi}{4}$ ,  $u = \frac{1}{\sqrt{2}}$ .

$$\frac{du}{dx} = \cos x.$$

$$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 \frac{du}{dx} dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 du = \left[ \frac{u^3}{3} \right]_0^{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{12}.$$

Q4b  $\operatorname{cosec} \theta + \cot \theta = \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta}$

$$= \frac{2 \cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}.$$

Q4ci Solve simultaneously to find  $R$ ,

$$x + py = 2ap + ap^3 \dots\dots(1)$$

$$x + qy = 2aq + aq^3 \dots\dots(2) \text{ where } p \neq q.$$

$$(2) - (1), \quad qy - py = 2aq - 2ap + aq^3 - ap^3,$$

$$(q - p)y = 2a(q - p) + a(q^3 - p^3),$$

$$(q - p)y = 2a(q - p) + a(q - p)(q^2 + qp + p^2),$$

$$(q - p)y = a(q - p)(2 + q^2 + qp + p^2).$$

$$\text{Hence } y = a(p^2 + pq + q^2 + 2) \dots\dots(3)$$

$$\text{Substitute (3) into (1), } x = 2ap + ap^3 - py$$

$$= 2ap + ap^3 - pa(p^2 + pq + q^2 + 2) = -apq(p + q).$$

$$\therefore R \text{ is } (-apq[p + q], a[p^2 + pq + q^2 + 2]).$$

Q4cii  $(0, a)$  satisfies  $y = \frac{1}{2}(p + q)x - apq$ ,  $\therefore a = -apq$  and  $pq = -1$ .

Q4ciii Since  $pq = -1$ ,  $\therefore R$  is  $(a[p + q], a[p^2 + pq + q^2 + 2])$ .

$$\therefore x = a(p + q) \text{ or } \frac{x}{a} = (p + q) \dots\dots(1) \text{ and}$$

$$y = a(p^2 + pq + q^2 + 2) = a(p^2 + 2pq + q^2 - pq + 2)$$

$$= a((p + q)^2 + 1 + 2),$$

$$\text{i.e. } y = a((p + q)^2 + 3) \dots\dots(2)$$

$$\text{Substitute (1) into (2), } y = a\left(\left(\frac{x}{a}\right)^2 + 3\right), \therefore y = \frac{1}{a}x^2 + 3a.$$

Q4d For  $n = 2$ ,  $4^n - 1 - 7n = 4^2 - 1 - 7 \times 2 = 1 > 0$ .

Assume  $4^n - 1 - 7n > 0$  is true for  $n = k > 2$ ,

i.e.  $4^k - 1 - 7k > 0$ , then for  $n = k + 1$ ,

$$4^{k+1} - 1 - 7(k + 1) = 4 \times 4^k - 1 - 7k - 7$$

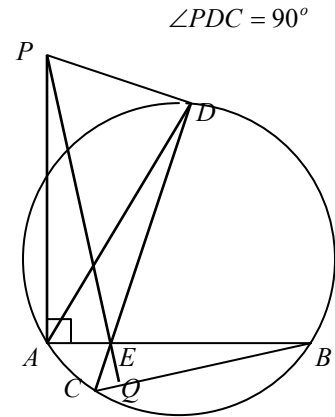
$$= 4(4^k - 1 - 7k) + 21k - 4 > 0. \text{ Hence } 4^n - 1 - 7n > 0 \text{ for all } n \geq 2.$$

Q5a  $V = \int_0^{\frac{\pi}{8}} \pi \sin^2 2x dx = \int_0^{\frac{\pi}{8}} \frac{\pi}{2} (1 - \cos 4x) dx = \left[ \frac{\pi}{2} \left( x - \frac{\sin 4x}{4} \right) \right]_0^{\frac{\pi}{8}}$

$$= \frac{\pi}{2} \left( \frac{\pi}{8} - \frac{1}{4} \right) = \frac{\pi(\pi - 2)}{16}.$$

Q5bi Quadrilateral  $DPAE$  is cyclic because the sum of the opposite angles is  $180^\circ$ .

Q5bii



$\angle APE = \angle ADE$ , because they are subtended by the same arc  $AE$  of the circle that passes through the vertices  $D, P, A$  and  $E$ . The circle is not shown in the above diagram.

$\angle ADC = \angle ABC$ , because they are subtended by the same arc  $AC$  of the circle shown above. Since  $\angle ADE$  and  $\angle ADC$  are the same angle,  $\therefore \angle APE = \angle ABC$ .

Q5biii Consider  $\triangle APE$  and  $\triangle QBE$ .

Since  $\angle APE = \angle ABC = \angle QBE$  and  $\angle AEP = \angle QEB$  (vertically opposite angles are equal),  $\therefore$  the third pair of angles must be the same, i.e.  $\angle EQB = \angle EAP = 90^\circ$ .  $\therefore PQ \perp BC$ .

Q5ci Let  $\sqrt{3} \sin 3t - \cos 3t = R \sin(3t - \alpha)$ , re-express the RHS to obtain  $\sqrt{3} \sin 3t - \cos 3t = R \sin 3t \cos \alpha - R \cos 3t \sin \alpha$ .

Compare the two sides,  $R \cos \alpha = \sqrt{3}$  and  $R \sin \alpha = 1$ .

Hence  $\tan \alpha = \frac{1}{\sqrt{3}}$  and  $R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 4$ .

$$\therefore \alpha = \frac{\pi}{6} \text{ and } R = 2 \therefore \sqrt{3} \sin 3t - \cos 3t = 2 \sin \left( 3t - \frac{\pi}{6} \right).$$

Q5cii  $x = 5 + \sqrt{3} \sin 3t - \cos 3t = 5 + 2 \sin \left( 3t - \frac{\pi}{6} \right)$ , hence the particle oscillates about  $x = 5$ , the centre of motion, with an amplitude of 2 units.

Q5ciii Maximum speed occurs when the particle passes

through the centre of motion, where  $\sin \left( 3t - \frac{\pi}{6} \right) = 0$ ,

$$3t - \frac{\pi}{6} = 0, \quad t = \frac{\pi}{18}.$$

Q6ai Binomial distribution:  $n = 5, p = \frac{2}{3}$ .

To earn one point Megan needs to pick 3 or more.

$$\Pr(X \geq 3) = {}^5C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 + {}^5C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1 + {}^5C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^0 = 0.7901$$

Q6aii Binomial distribution:  $n = 18, p = 0.7901$ .

$$\Pr(X = 18) = 0.7901^{18} = 0.01.$$

Q6aiii Binomial distribution:  $n = 18, p = 0.7901, q = 0.2099$ .

$$\Pr(X \leq 16) = 1 - \Pr(X = 17) - \Pr(X = 18) \\ = 1 - {}^{18}C_{17} (0.7901)^{17} (0.2099)^1 - 0.01 = 0.92.$$

Q6bi Maximum height is reached when  $\frac{dy}{dt} = 0$ .

$$y = -4.9t^2 + 200t + 5000, \frac{dy}{dt} = -9.8t + 200 = 0, \therefore t = 20.4 \text{ s} \\ \text{and } y = 7040.8 \text{ m.}$$

Q6bii  $x = 200t, \therefore \frac{dx}{dt} = 200$ .

Descending slope is  $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{-9.8t + 200}{200}$ .

At  $45^\circ, \frac{dy}{dx} = \tan(-45^\circ) = -1, \therefore \frac{-9.8t + 200}{200} = -1, t = 40.8 \text{ s.}$

At  $60^\circ, \frac{dy}{dx} = \tan(-60^\circ) = -\sqrt{3},$

$$\therefore \frac{-9.8t + 200}{200} = -\sqrt{3}, t = 55.8 \text{ s.}$$

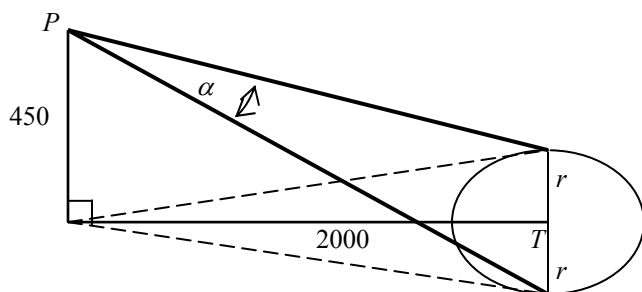
Earliest time is 40.8 s and the latest time is 55.8 s.

Q6biii The latest time is when the speed =  $350 \text{ ms}^{-1}$ ,

i.e.  $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 350, \therefore 200^2 + (200 - 9.8t)^2 = 122500,$   
 $t = 49.7 \text{ s.}$

Q7ai  $PT = \sqrt{450^2 + 2000^2} = 2050 \text{ m}$

$$\frac{r}{2050} = \tan(0.05), \therefore r = 102.6 \text{ m.}$$



Q7aii  $r = 2050 \tan\left(\frac{\alpha}{2}\right), \therefore \frac{dr}{d\alpha} = 1025 \sec^2\left(\frac{\alpha}{2}\right).$

Given  $\frac{d\alpha}{dt} = 0.02$  and  $\alpha = 0.1,$

$$\therefore \frac{dr}{dt} = \frac{dr}{d\alpha} \times \frac{d\alpha}{dt} = 20.5 \sec^2(0.05) = 20.6 \text{ m per hour.}$$

Q7bi  $f(x) = Ax^3 - Ax + 1, \text{ where } A > 0.$

$$f'(x) = 3Ax^2 - A. \therefore f'\left(\pm \frac{\sqrt{3}}{3}\right) = 3A\left(\pm \frac{\sqrt{3}}{3}\right)^2 - A = 0.$$

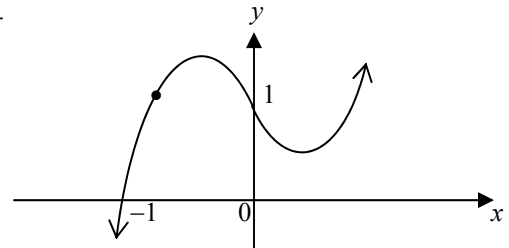
$$\therefore f(x) \text{ has stationary points at } x = \pm \frac{\sqrt{3}}{3}.$$

Q7bii At  $x = \frac{\sqrt{3}}{3}$ , the value of the function is a minimum,

$$y = f\left(\frac{\sqrt{3}}{3}\right) = A\left(\frac{\sqrt{3}}{3}\right)^3 - A\left(\frac{\sqrt{3}}{3}\right) + 1 = -\frac{2A}{3\sqrt{3}} + 1.$$

For the local minimum to be a positive value,  $-\frac{2A}{3\sqrt{3}} + 1 > 0,$

$$\therefore A < \frac{3\sqrt{3}}{2}.$$



$$\therefore f(x) \text{ has exactly one zero when } A < \frac{3\sqrt{3}}{2}.$$

Q7biii Since  $f(-1) = 1$  and  $f'(-1) = 2A > 0$ , given  $A > 0$ , the only zero must be at  $x < -1$ .  $\therefore f(x)$  does not have a zero in the interval  $-1 \leq x \leq 1$  when  $0 < A < \frac{3\sqrt{3}}{2}$ .

Q7biv  $g(\theta) = 2 \cos \theta + \tan \theta, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

$$g'(\theta) = -2 \sin \theta + \sec^2 \theta = -2 \sin \theta + \frac{1}{\cos^2 \theta}$$

$$= \frac{-2 \sin \theta \cos^2 \theta + 1}{\cos^2 \theta} = \frac{-2 \sin \theta (1 - \sin^2 \theta) + 1}{\cos^2 \theta}$$

$$= \frac{2 \sin^3 \theta - 2 \sin \theta + 1}{\cos^2 \theta}. \text{ Since } 0 < 2 < \frac{3\sqrt{3}}{2} \text{ and } -1 < \sin \theta < 1$$

$$\therefore -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \therefore 2 \sin^3 \theta - 2 \sin \theta + 1 \text{ has no zeros. Hence}$$

$g(\theta)$  does not have any stationary points.

Q7bv  $\therefore g(\theta)$  must be a one-to-one function in the interval

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \therefore g(\theta) \text{ has an inverse function.}$$

Please inform [mathline@itute.com](mailto:mathline@itute.com) re conceptual, mathematical and/or typing errors.