# Algebra

**The binomial theorem**

This theorem is used to expand expression of the form $(p+q)^n$, where $n$ is a positive integer.

$(p+q)^n = {}^nC_0 p^n q^0 + {}^nC_1 p^{n-1} q^1 + {}^nC_2 p^{n-2} q^2 + \dots\dots + {}^nC_n p^0 q^n$

Example 1 Expand $(2x-3)^4$.

$$(2x-3)^4 = {}^4C_0(2x)^4(-3)^0 + {}^4C_1(2x)^3(-3)^1 + {}^4C_2(2x)^2(-3)^2 + {}^4C_3(2x)^1(-3)^3 + {}^4C_4(2x)^0(-3)^4$$
$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Example 2 Find the coefficient of the term containing $x^6$ in the expansion of $(3x-a)^9$, where $a$ is a constant.

The required term is ${}^9C_3(3x)^6(-a)^3$.
The coefficient is ${}^9C_3(3)^6(-a)^3 = -61236a^3$.

**Factorisation of polynomial functions**

Not all polynomials can be factorised analytically (by algebraic methods). The following methods are applicable to those polynomials that have factors involving rational numbers and in some cases, surds.

*Common factors* Look for common factors and separate them from the terms of the polynomial by means of the distributive law in reverse. This is always the first step in all factorisations.

Example Factorise $f(x) = 25x^3 - 10x^2 + \frac{5}{2}x$.

$f(x) = 25x^3 - 10x^2 + \frac{5}{2}x = (5)(5)xxx - (5)(2)xx + (5)\left(\frac{1}{2}\right)x$. All three terms have common factors 5 and $x$.

Hence $f(x) = 5x\left(5xx - 2x + \frac{1}{2}\right) = 5x\left(5x^2 - 2x + \frac{1}{2}\right)$.

*Difference of two squares* Rewrite the polynomial in the form $a^2 - b^2$ that can be factorised as $(a-b)(a+b)$.

Note: Polynomials in the form of *sum of two squares* cannot be factorised.

Example 1 Factorise $36x^2 - \frac{1}{25}$.

The polynomial has no common factors.

$$36x^2 - \frac{1}{25} = (6x)^2 - \left(\frac{1}{5}\right)^2 = \left(6x - \frac{1}{5}\right)\left(6x + \frac{1}{5}\right).$$

Example 2 Factorise $y = 27x^3 - 12x$.

Separate the common factors 3 and $x$ from the two terms, rewrite the resulting two terms as difference of two squares and then factorise.

$$y = 3x(9x^2 - 4) = 3x\left((3x)^2 - 2^2\right) = 3x(3x-2)(3x+2).$$

Example 3 Factorise $7x^2 - 196$.

Separate the common factor 7 from the two terms, rewrite the resulting two terms as difference of two squares and then factorise.

$$7x^2 - 196 = 7(x^2 - 28) = 7\left(x^2 - \left(2\sqrt{7}\right)^2\right) = 7\left(x - 2\sqrt{7}\right)\left(x + 2\sqrt{7}\right).$$

Example 4 Factorise $x^4 - 9$ over Q.

Rewrite the binomial as difference of two squares and then factorise.

$$x^4 - 9 = (x^2)^2 - 3^2 = (x^2 - 3)(x^2 + 3)$$
$$= \left(x^2 - \left(\sqrt{3}\right)^2\right)(x^2+3) = \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)(x^2+3)$$

Example 5 Factorise $(2x-3)^2 - 5$ over R.

Do not expand. Express the polynomial as difference of two squares and then factorise.

$$(2x-3)^2 - 5 = (2x-3)^2 - \left(\sqrt{5}\right)^2 = \left(2x - 3 - \sqrt{5}\right)\left(2x - 3 + \sqrt{5}\right).$$

*Quadratic trinomials* For trinomials that have linear factors involving rational numbers, the quickest method is by trial and error.

$x^2 + bx + c = (x+p)(x+q)$, by trial and error, find numbers $p$ and $q$ such that $pq = c$ AND $p + q = b$.

$ax^2 + bx + c = (mx+p)(nx+q)$, by trial and error, find numbers $m$, $n$, $p$ and $q$ such that $mn = a$, $pq = c$ AND $mq + np = b$.

Note: Check the value of $b^2 - 4c$ in the first case and $b^2 - 4ac$ in the second case before trying. If the value is negative, the trinomial has no linear factors. If the value is zero, the two linear factors are the same.

Example 1 Factorise $3x^2 - 2x + 1$.

This trinomial cannot be factorised because $b^2 - 4ac = (-2)^2 - 4(3)(1) = -8$, a negative value.

Example 2 Factorise $f(x) = -2x^2 + 4x + 6$.

The trinomial has a common factor of $-2$, hence $f(x) = -2(x^2 - 2x - 3)$.
For the resulting trinomial inside the brackets, $b^2 - 4c = (-2)^2 - 4(-3) = 16$, a positive value; it can be factorised.
$f(x) = -2(x^2 - 2x - 3) = -2(x-3)(x+1)$, by trial and error.

Example 3 Factorise $y = 2rx^2 - 28rx + 98r$.

$y = 2rx^2 - 28rx + 98r = 2r(x^2 - 14x + 49)$ Common factors.

For the resulting trinomial, $b^2 - 4c = (-14)^2 - 4(49) = 0$, hence the two linear factors are the same.

$\therefore y = 2r(x-7)^2$.

Example 4 Factorise $x^4 - x^2 - 12$ completely over Q.

Rewrite the polynomial as $(x^2)^2 - (x^2) - 12$

$= (x^2 - 4)(x^2 + 3)$ Trial and error

$= (x-2)(x+2)(x^2+3)$ Difference of two squares

Example 5 Factorise $(x+3)^2 - 2(x+3) - 35$.

Do not expand. Replace $x+3$ by *y*, then factorise by trial and error.

$(x+3)^2 - 2(x+3) - 35 = y^2 - 2y - 35 = (y-7)(y+5)$

$= (x+3-7)(x+3+5) = (x-4)(x+8)$

*Completing the square* If you fail to find the linear factors of a quadratic trinomial by trial and error, try completing the square.

Example 1 Factorise $x^2 - 6x + 2$ over R.

$x^2 - 6x + 2 = x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(\frac{-6}{2}\right)^2 + 2$

$= x^2 - 6x + 9 - 9 + 2 = (x^2 - 6x + 9) - 7$

$= (x-3)^2 - (\sqrt{7})^2 = (x - 3 - \sqrt{7})(x - 3 + \sqrt{7})$.

Example 2 Factorise $2x^2 - 3x - 6$ over R.

$2x^2 - 3x - 6 = 2\left[x^2 - \frac{3}{2}x - 3\right] = 2\left[x^2 - \frac{3}{2}x + \left(\frac{1}{2} \times \frac{3}{2}\right)^2 - \left(\frac{3}{4}\right)^2 - 3\right]$

$= 2\left[\left(x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2\right) - \left(\frac{3}{4}\right)^2 - 3\right] = 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{57}{16}\right]$

$= 2\left[\left(x - \frac{3}{4}\right)^2 - \left(\frac{\sqrt{57}}{4}\right)^2\right] = 2\left[\left(x - \frac{3}{4}\right) - \frac{\sqrt{57}}{4}\right]\left[\left(x - \frac{3}{4}\right) + \frac{\sqrt{57}}{4}\right]$

$= \left(x - \frac{3}{4} - \frac{\sqrt{57}}{4}\right)\left(x - \frac{3}{4} + \frac{\sqrt{57}}{4}\right)$.

Example 3 Factorise $x^4 - 2x^2 - 7$ over R.

$x^4 - 2x^2 - 7 = (x^2)^2 - 2(x^2) - 1^2 + 1^2 - 7$

$= ((x^2)^2 - 2(x^2) - 1) - 6 = (x^2 - 1)^2 - (\sqrt{6})^2$

$= (x^2 - 1 - \sqrt{6})(x^2 - 1 + \sqrt{6})$

$= (x^2 - (1 + \sqrt{6}))(x^2 - (1 - \sqrt{6}))$

$= \left(x^2 - \left(\sqrt{1+\sqrt{6}}\right)^2\right)\left(x^2 - \left(\sqrt{1-\sqrt{6}}\right)^2\right)$

$= \left(x - \sqrt{1+\sqrt{6}}\right)\left(x + \sqrt{1+\sqrt{6}}\right)\left(x - \sqrt{1-\sqrt{6}}\right)\left(x + \sqrt{1-\sqrt{6}}\right)$.

Example 4 Factorise $x^4 - x^2 + 16$ over R.

Add $9x^2$ to complete the square.

$x^4 - x^2 + 16 = (x^2)^2 - x^2 + 16 + 9x^2 - 9x^2$

$= ((x^2)^2 + 8x^2 + 16) - 9x^2$

$= (x^2 + 4)^2 - (3x)^2 = (x^2 - 3x + 4)(x^2 + 3x + 4)$.

The two quadratic factors do not have linear factors, $\because b^2 - 4ac < 0$.

Note: Examples 3 and 4 show two methods in completing the square for polynomials of the form $(x^2)^2 - Bx^2 + C$.

If $B^2 - 4C > 0$, follow the method shown in example 3, i.e. add and subtract a constant to complete the square.

If $B^2 - 4C < 0$, follow the method shown in example 4, i.e. add and subtract a $x^2$ term to complete the square.

*Sum/difference of two cubes* Both forms can be factorised to a linear factor and a quadratic factor. The quadratic factor cannot be factorised further.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example 1 Factorise $2x^3 - 54$ over R.

The two terms in the cubic function have a common factor of 2.

$2x^3 - 54 = 2(x^3 - 27) = 2(x^3 - 3^3) = 2(x-3)(x^2 + 3x + 9)$.

Example 2 Factorise $625x^3 + 40$ over R.

$625x^3 + 40 = 5(125x^3 + 8) = 5((5x)^3 + 2^3)$

$= 5(5x+2)((5x)^2 - 2(5x) + 2^2) = 5(5x+2)(25x^2 - 10x + 4)$.

Example 3 Factorise $1000x^3 + 5^{\frac{3}{2}}$ over R.

$1000x^3 + 5^{\frac{3}{2}} = (10x)^3 + (\sqrt{5})^3$

$= (10x + \sqrt{5})\left((10x)^2 - (10x)(\sqrt{5}) + (\sqrt{5})^2\right)$

$= (10x + \sqrt{5})(100x^2 - 10\sqrt{5}x + 5)$

$= 5(10x + \sqrt{5})(20x^2 - 2\sqrt{5}x + 1)$.

Example 4 Factorise $(2x-1)^3 + 1$ over R.

Do not expand, use 'the sum of two cubes'.

$(2x-1)^3 + 1 = (2x-1)^3 + 1^3$

$= ((2x-1) + 1)((2x-1)^2 - 1(2x-1) + 1^2)$

$= (2x)(4x^2 - 4x + 1 - 2x + 1 + 1) = 2x(4x^2 - 6x + 3)$.

Example 5   Factorise $x^6 - 64$ over R.

Treat $x^6 - 64$ as difference of two squares.

$$x^6 - 64 = (x^3)^2 - (2^3)^2 = (x^3 - 2^3)(x^3 + 2^3)$$
$$= (x-2)(x^2 + 2x + 4)(x+2)(x^2 - 2x + 4).$$

If $x^6 - 64$ is treated as difference of two cubes, then

$$x^6 - 64 = (x^2)^3 - (2^2)^3 = (x^2 - 2^2)\left((x^2)^2 + 2^2 x^2 + (2^2)^2\right)$$
$$= (x-2)(x+2)\left((x^2)^2 + 4x^2 + 16\right)$$
$$= (x-2)(x+2)\left((x^2)^2 + 4x^2 + 16 + 4x^2 - 4x^2\right)$$
$$= (x-2)(x+2)\left((x^2)^2 + 8x^2 + 16 - 4x^2\right)$$
$$= (x-2)(x+2)\left((x^2+4)^2 - (2x)^2\right)$$
$$= (x-2)(x+2)(x^2 + 4 - 2x)(x^2 + 4 + 2x).$$

*Grouping* The four terms in a cubic polynomial are grouped into two and two. The two groups are then factorised. Correct grouping gives a common factor in the two groups.

Example 1   Factorise $2x^3 - 3x^2 - 32x + 48$ over R.

Group the first two terms and the last two.

$$2x^3 - 3x^2 - 32x + 48 = (2x^3 - 3x^2) - (32x - 48)$$
$$= x^2(2x-3) - 16(2x-3) = (2x-3)(x^2 - 16)$$
$$= (2x-3)(x^2 - 4^2) = (2x-3)(x-4)(x+4).$$

Note: It is also possible to proceed by grouping the first and the third terms; and the second and the last terms.

Example 2   Factorise $8x^3 - 2x^2 + x - 1$ over R.

$$8x^3 - 2x^2 + x - 1 = (8x^3 - 1) - (2x^2 - x) = \left((2x)^3 - 1^3\right) - (2x^2 - x)$$
$$= (2x-1)\left((2x)^2 + (2x)1 + 1^2\right) - x(2x-1)$$
$$= (2x-1)(4x^2 + 2x + 1) - x(2x-1)$$
$$= (2x-1)(4x^2 + 2x + 1 - x) = (2x-1)(4x^2 + x + 1).$$

*The factor theorem*

Consider a polynomial $P(x)$ that has $x - \alpha$ as a linear factor. Then $P(x) = (x-\alpha)Q(x)$, where $Q(x)$ is a polynomial one degree lower than $P(x)$, obtained by expanding and comparing coefficients, or dividing $P(x)$ by $x - \alpha$.
Replacing $x$ by $\alpha$ in $P(x) = (x-\alpha)Q(x)$,
$P(\alpha) = (\alpha - \alpha)Q(\alpha)$. Hence $P(\alpha) = 0$.

Conversely, for any polynomial $P(x)$, if $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$. This statement is known as the factor theorem, and can be used to find the linear factors of a polynomial if other methods failed.

The factor theorem is best used for polynomials with linear factors of rational coefficients. The value(s) of $\alpha$ is found by trial and error. The possible values of $\alpha$ for trying depend on the first and last coefficients of

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n, \quad \alpha = \pm\frac{a.factor.of.a_n}{a.factor.of.a_0}.$$

If all these $\alpha$ values give $P(\alpha) \neq 0$, it does not necessarily mean that $P(x)$ has no linear factors, because the coefficients of the linear factor(s) may be irrational.

Example 1   Use the factor theorem to find a linear factor of $3x^3 - 2x^2 - 7x - 2$, then find the quadratic factor and hence all the linear factors.

Let $P(x) = 3x^3 - 2x^2 - 7x - 2$. The possible values of $\alpha$ for testing are $\alpha = \pm\frac{1,2}{1,3}$, i.e. $\alpha = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2$.

$P(1) = 3(1)^3 - 2(1)^2 - 7(1) - 2 \neq 0$.

$P(-1) = 3(-1)^3 - 2(-1)^2 - 7(-1) - 2 = 0$, $\therefore x + 1$ is a factor.

Divide $P(x)$ by $x + 1$ to find the quadratic factor.

$$\begin{array}{r l}
 & 3x^2 - 5x - 2 \\
x+1\,) & 3x^3 - 2x^2 - 7x - 2 \\
 & -(3x^3 + 3x^2) \\
\hline
 & -5x^2 - 7x \\
 & -(-5x^2 - 5x) \\
\hline
 & -2x - 2 \\
 & -(-2x - 2) \\
\hline
 & 0
\end{array}$$

Hence $P(x) = (x+1)(3x^2 - 5x - 2) = (x+1)(3x+1)(x-2)$.

Another possible outcome:

$P\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right)^3 - 2\left(-\frac{1}{3}\right)^2 - 7\left(-\frac{1}{3}\right) - 2 = 0$, $\therefore x + \frac{1}{3}$ is a factor.

$$\begin{array}{r l}
 & 3x^2 - 3x - 6 \\
x+\frac{1}{3}\,) & 3x^3 - 2x^2 - 7x - 2 \\
 & -(3x^3 + x^2) \\
\hline
 & -3x^2 - 7x \\
 & -(-3x^2 - x) \\
\hline
 & -6x - 2 \\
 & -(-6x - 2) \\
\hline
 & 0
\end{array}$$

Hence $P(x) = \left(x + \frac{1}{3}\right)(3x^2 - 3x - 6) = 3\left(x + \frac{1}{3}\right)(x^2 - x - 2)$

$= 3\left(x + \frac{1}{3}\right)(x-2)(x+1)$. It is equivalent to the previous result.

Example 2 Given $x+1$ is a factor of $3x^4+x^3-9x^2-9x-2$, find the cubic factor $Q(x)$ such that

$$3x^4+x^3-9x^2-9x-2=(x+1)Q(x).$$

Let $Q(x)=ax^3+bx^2+cx+d$, then
$3x^4+x^3-9x^2-9x-2=(x+1)(ax^3+bx^2+cx+d)$.
Expand to obtain $3x^4+x^3-9x^2-9x-2$
$=ax^4+(a+b)x^3+(b+c)x^2+(c+d)x+d$.
Compare the coefficients on both sides, $a=3$, $a+b=1$, $b+c=-9$, $c+d=-9$ and $d=-2$. Hence $b=-2$, $c=-7$.
$\therefore Q(x)=3x^3-2x^2-7x-2$.
This example illustrates an alternative method in finding $Q(x)$ to long division shown in the last example.

Example 3 Use the factor theorem to find the linear factors of $x^4-x^3+x^2-3x+2$.

Let $P(x)=x^4-x^3+x^2-3x+2$. Test $\alpha=\pm1,\pm2$.
$P(-1)=(-1)^4-(-1)^3+(-1)^2-3(-1)+2\neq0$
$P(1)=(1)^4-(1)^3+(1)^2-3(1)+2=0$, $\therefore x-1$ is a factor.
Hence $P(x)=(x-1)Q(x)$.
Use long division or comparing coefficients to find $Q(x)=x^3+x-2$.
Hence $P(x)=(x-1)(x^3+x-2)$. Use the factor theorem on $Q(x)=x^3+x-2$ to find its linear factor. Test $\alpha=\pm1,\pm2$.
$Q(1)=(1)^3+(1)-2=0$, $\therefore x-1$ is a factor of $Q(x)$.
Hence $P(x)=(x-1)(x-1)T(x)$. $T(x)$ is quadratic and found by long division of $P(x)$ by the expansion of $(x-1)(x-1)$, or comparison of coefficients as discussed in example 2,
$T(x)=x^2+x+2$.
$\therefore P(x)=(x-1)(x-1)(x^2+x+2)$.

**The remainder theorem** When a polynomial $P(x)$ is divided by a linear binomial $\alpha x-\beta$, the remainder can be found quickly without actually carrying out the division.

Since $\qquad Q(x) \qquad$ Quotient

$\alpha x-\beta\,\big)\,P(x) \qquad$ Dividend (Polynomial)

Divisor $\qquad R \qquad$ Remainder

$\therefore P(x)=(\alpha x-\beta)Q(x)+R$.
When $x=\dfrac{\beta}{\alpha}$, $P\left(\dfrac{\beta}{\alpha}\right)=\left(\alpha\left(\dfrac{\beta}{\alpha}\right)-\beta\right)Q(x)+R=R$,
i.e. the remainder $R=P\left(\dfrac{\beta}{\alpha}\right)$ when $P(x)$ is divided by $\alpha x-\beta$.
This is known as the remainder theorem.

Example 1 Find the remainder when $P(x)=2x^4-x^3+5x-11$ is divided by (i) $x+5$, (ii) $2x-3$, (iii) $x-2a$.

(i) $R=P(-5)=2(-5)^4-(-5)^3+5(-5)-11=1339$

(ii) $R=P\left(\dfrac{3}{2}\right)=2\left(\dfrac{3}{2}\right)^4-\left(\dfrac{3}{2}\right)^3+5\left(\dfrac{3}{2}\right)-11=\dfrac{13}{4}$

(iii)
$R=P(2a)=2(2a)^4-(2a)^3+5(2a)-11=32a^4-8a^3+10a-11$

Example 2 Given that $x-2$ is a factor of $3x^3+px^2+qx-2$ and the remainder is $-20$ when the cubic polynomial is divided by $x+2$. Find the values of *p* and *q*.

Use the factor theorem and the remainder theorem to set up two simultaneous equations for *p* and *q*.
Let $P(x)=3x^3+px^2+qx-2$.
$x-2$ is a factor of $P(x)$, $\therefore P(2)=0$,
$\therefore 3(2)^3+p(2)^2+q(2)-2=0$, $\qquad \therefore 2p+q=-11$ ……(1)
The remainder is $-20$ when divided by $x+2$, $\therefore P(-2)=-20$,
$\therefore 3(-2)^3+p(-2)^2+q(-2)-2=-20$, $\qquad \therefore 2p-q=3$ ……(2)

Solve eqs (1) and (2) for *p* and *q*.
(1) + (2), $4p=-8$, $\therefore p=-2$
(1) – (2), $2q=-14$, $\therefore q=-7$

**Exponential (index) laws**

1. $a^m\times a^n=a^{m+n}$
2. $\dfrac{1}{a^n}=a^{-n}$; $a^m=\dfrac{1}{a^{-m}}$; $\dfrac{a^m}{a^n}=a^{m-n}$
3. $(a^m)^n=a^{mn}$
4. $(ab)^n=a^nb^n$

Good to know: $a^0=1$, $a^{\frac{1}{p}}=\sqrt[p]{a}$, $a^{\frac{q}{p}}=\left(\sqrt[p]{a}\right)^q$ or $\sqrt[p]{a^q}$.

Example 1 Simplify $\dfrac{(ab)^{2n}-b^{3n}}{(ab^4)^n+b^{4n}}$, and express with positive indices.

$$\frac{(ab)^{2n}-b^{4n}}{(ab^4)^n+b^{5n}}=\frac{a^{2n}b^{2n}-b^{4n}}{a^nb^{4n}+b^{5n}}=\frac{b^{2n}(a^{2n}-b^{2n})}{b^{4n}(a^n+b^n)}$$

$$=\frac{b^{-2n}\left((a^n)^2-(b^n)^2\right)}{(a^n+b^n)}=\frac{b^{-2n}(a^n-b^n)(a^n+b^n)}{(a^n+b^n)}$$

$$=\frac{a^n-b^n}{b^{2n}}.$$

Example 2 Simplify $\dfrac{e^{2x+1}-4e^{x+1}+3e}{e^{x+1}-e}$.

$$\frac{e^{2x+1}-4e^{x+1}+3e}{e^{x+1}-e}=\frac{e\left(e^{2x}-4e^{x}+3\right)}{e\left(e^{x}-1\right)}=\frac{e^{0}\left(\left(e^{x}\right)^{2}-4\left(e^{x}\right)+3\right)}{\left(e^{x}-1\right)}$$
$$=\frac{\left(e^{x}-3\right)\left(e^{x}-1\right)}{\left(e^{x}-1\right)}=e^{x}-3 \text{ for } x\neq 0.$$

Note: It is necessary to state that $x\neq 0$, $\because \dfrac{e^{2x+1}-4e^{x+1}+3e}{e^{x+1}-e}$ is undefined for $x=0$ whilst $e^{x}-3$ is defined for all $x$, $\therefore$ they cannot be equal at $x=0$.

Example 3 Simplify $\dfrac{p^{\frac{5}{2}}+2p^{\frac{3}{2}}}{5p^{\frac{3}{2}}+10p^{\frac{1}{2}}}$.

$$\frac{p^{\frac{5}{2}}+2p^{\frac{3}{2}}}{5p^{\frac{3}{2}}+10p^{\frac{1}{2}}}=\frac{p^{\frac{3}{2}}(p+2)}{5p^{\frac{1}{2}}(p+2)}=\frac{p^{\frac{3}{2}-\frac{1}{2}}}{5}=\frac{p}{5}.$$

Example 4 Simplify $f(n)=\left(100^{n}+2\times 10^{n+1}+100\right)^{\frac{3}{2}}$. Show that $f(2)=1000f(0)$.

$$\left(100^{n}+2\times 10^{n+1}+100\right)^{\frac{3}{2}}=\left(\left(10^{2}\right)^{n}+2\times 10\times 10^{n}+10^{2}\right)^{\frac{3}{2}}$$
$$=\left(\left(10^{n}\right)^{2}+2(10)\left(10^{n}\right)+10^{2}\right)^{\frac{3}{2}}=\left(\left(10^{n}+10\right)^{2}\right)^{\frac{3}{2}}=\left(10^{n}+10\right)^{3}$$
$$\therefore f(n)=\left(10^{n}+10\right)^{3}.$$
$$f(0)=\left(10^{0}+10\right)^{3}=11^{3};$$
$$f(2)=\left(10^{2}+10\right)^{3}=(110)^{3}=(10\times 11)^{3}=10^{3}\times 11^{3}=1000f(0).$$

Example 5 Simplify $\dfrac{6\times\sqrt[3]{2x^{5}y^{2}}}{\sqrt{\left(6x^{6}y\right)^{\frac{2}{3}}}}$, express in positive indices.

$$\frac{6\times\sqrt[3]{2x^{5}y^{2}}}{\sqrt{\left(6x^{6}y\right)^{\frac{2}{3}}}}=\frac{6\left(2x^{5}y^{2}\right)^{\frac{1}{3}}}{\left(\left(6x^{6}y\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}}=\frac{2\times 3\times 2^{\frac{1}{3}}x^{\frac{5}{3}}y^{\frac{2}{3}}}{\left(2\times 3x^{6}y\right)^{\frac{1}{3}}}$$
$$=\frac{3\times 2^{\frac{4}{3}}x^{\frac{5}{3}}y^{\frac{2}{3}}}{2^{\frac{1}{3}}\times 3^{\frac{1}{3}}x^{2}y^{\frac{1}{3}}}=2\times 3^{\frac{2}{3}}x^{-\frac{1}{3}}y^{\frac{1}{3}}=\frac{2\times 3^{\frac{2}{3}}y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

**Logarithm laws**

For $p,q>0$,

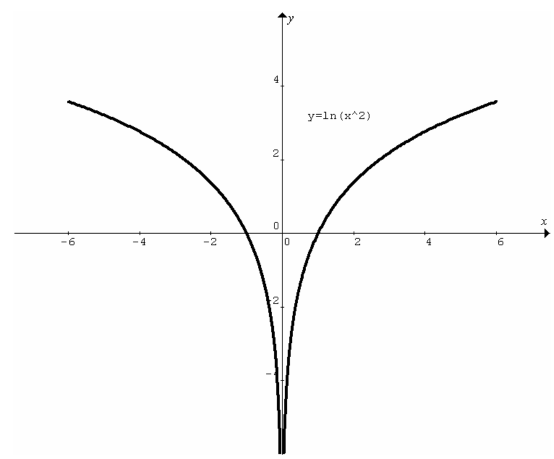
1. $\log_a p+\log_a q=\log_a(pq)$
2. $\log_a p-\log_a q=\log_a\left(\dfrac{p}{q}\right)$; $-\log_a q=\log_a\left(\dfrac{1}{q}\right)$
3. $\log_a p^{n}=n\log_a p$
4. $\log_b p=\dfrac{\log_a p}{\log_a b}$

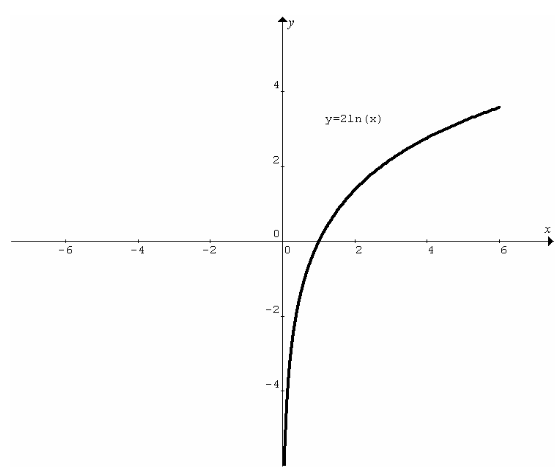
Law 4 shows the relationship between $\log_a p$ and $\log_b p$.

Good to remember: $\log_a 1=0$; $\log_a a=1$;
$\log_a p$ is undefined for $p\leq 0$;
$\log_a p<0$ for $0<p<1$; $\log_a p>0$ for $p>1$.

For even $n$, $\log_a p^{n}$ is defined for all $p\in R$ whilst $n\log_a p$ is defined only for $p>0$,
$\therefore \log_a p^{n}=n\log_a p$ for $p>0$, and
$\log_a p^{n}\neq n\log_a p$ for $p\leq 0$.

The following graphs of $y=\log_e x^{2}$ and $y=2\log_e x$ illustrate the point.



Example 1 Evaluate $5\log_2\left(\frac{1}{32}\right)$.

$5\log_2\left(\frac{1}{32}\right)=5\log_2\left(\frac{1}{2^5}\right)=5\log_2 2^{-5}=-5\times 5\log_2 2=-25$ .

Example 2 Simplify $2\log_{10}\left(3x^2y\right)-3\log_{10}\left(2xy^2\right)$.

$2\log_{10}\left(3x^2y\right)-3\log_{10}\left(2xy^2\right)=\log_{10}\left(3x^2y\right)^2-\log_{10}\left(2xy^2\right)^3$

$=\log_{10}\left(9x^4y^2\right)-\log_{10}\left(8x^3y^6\right)=\log_{10}\left(\frac{9x^4y^2}{8x^3y^6}\right)=\log_{10}\left(\frac{9x}{8y^4}\right)$.

Example 3 Evaluate $\log_3 10$ .

Your calculator has only log (i.e. $\log_{10}$ ) and ln (i.e. $\log_e$ ).

$\log_3 10=\frac{\log_{10}10}{\log_{10}3}=2.0959$ , or $\log_3 10=\frac{\log_e 10}{\log_e 3}=2.0959$ .

Example 4 Show that $3\log_4 x-2\log_8 x=\frac{5}{6}\log_2 x$ .

Change both logarithms on the left side of the identity to base 2.

$3\log_4 x-2\log_8 x=\frac{3\log_2 x}{\log_2 4}-\frac{2\log_2 x}{\log_2 8}$

$=\frac{3\log_2 x}{\log_2 2^2}-\frac{2\log_2 x}{\log_2 2^3}=\frac{3}{2}\log_2 x-\frac{2}{3}\log_2 x$

$=\left(\frac{3}{2}-\frac{2}{3}\right)\log_2 x=\frac{5}{6}\log_2 x$.

Example 5 Show that $\frac{1}{\log_e 10}=\log_{10} e$ .

Change both sides to a common base $b$.

$LHS=\frac{1}{\frac{\log_b 10}{\log_b e}}=\frac{\log_b e}{\log_b 10}$ .

$RHS=\frac{\log_b e}{\log_b 10}$ . $\therefore LHS=RHS$ .

Example 6 Given $10^p=e^q$, (i) find $q$ in terms of $p$, (ii) find $p$ in terms of $q$.

(i) $10^p=e^q$ , $\log_e 10^p=\log_e e^q$ , $p\log_e 10=q\log_e e$ ,
$\therefore q=p\log_e 10$ .

(ii) Since $q=p\log_e 10$ , $\therefore p=\frac{q}{\log_e 10}=q\log_{10} e$ .

These two results show the way to change the base of an exponential function.

$10^p=e^{p\log_e 10}$; $e^q=10^{q\log_{10} e}$ . In general, $a^x=b^{x\log_b a}$ .

Example 6 Change $5^x$ to base 10 and base $e$.

$5^x=10^{x\log_{10}5}\approx 10^{0.6990x}$; $5^x=e^{x\log_e 5}\approx e^{1.6094x}$ .

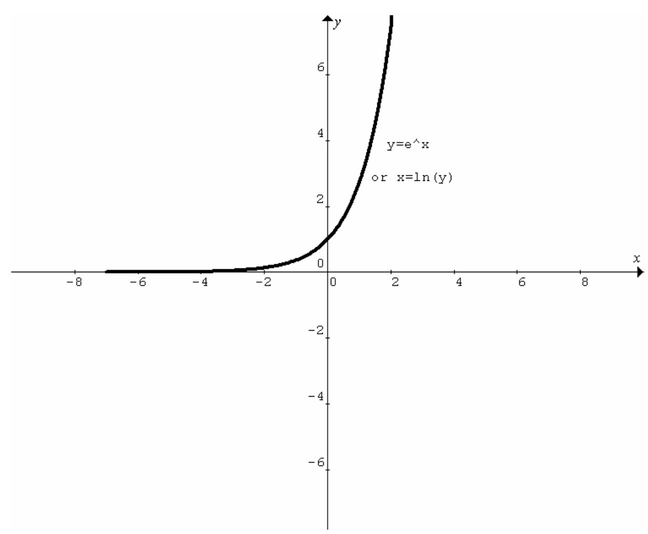
**Equivalent relations**

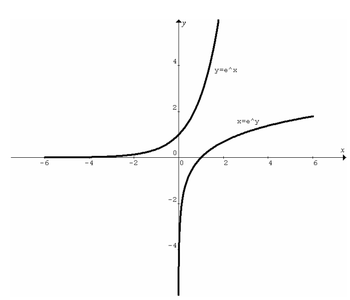
Examples are: $y=x^2 \Leftrightarrow x=\pm\sqrt{y}$ ;
$y=Sin(x)\Leftrightarrow x=Sin^{-1}(y)$;
$y=10^x \Leftrightarrow x=\log_{10} y$ ;
$y=e^x \Leftrightarrow x=\log_e y$ .

In each of the above cases, both left and right statements give exactly the same relationship between $x$ and $y$, i.e. they are equivalent. Try to plot the graphs of a pair of equivalent relations. They are the same plot. The left relation uses $y$ as the subject, and the right relation uses $x$. In the last two examples, the left relations are expressed in index (exponential) form whilst the right relations are in logarithm form.

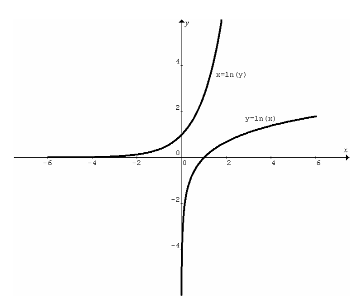
**Inverse relations**

$y=e^x$ and $x=\log_e y$ are equivalent relations, but
$y=e^x$ and $x=e^y$ are inverse relations, and so are
$y=\log_e x$ and $x=\log_e y$ . (Read each relation carefully)
In inverse relations, the x and y-coordinates of all the points are interchanged.



$y=e^x$ and $x=e^y$



$y=\log_e x$ and $x=\log_e y$

$y = e^x$ and $x = e^y$ are inverses of each other. Express $x = e^y$ with $y$ as the subject, $y = \log_e x$ .

$y = \log_e x$ and $x = \log_e y$ are inverses of each other. Express $x = \log_e y$ with $y$ as the subject, $y = e^x$ .

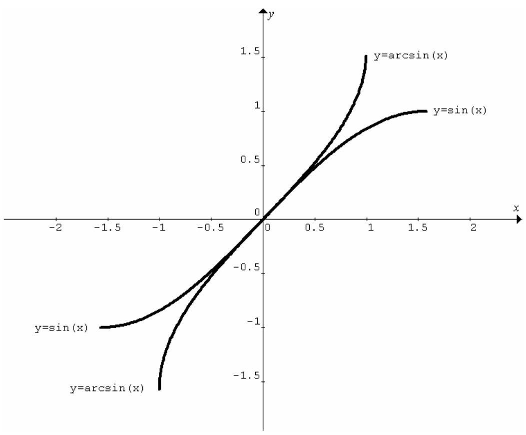
Hence $y = e^x$ and $y = \log_e x$ are inverses of each other, as discussed previously in Functions and Graphs.

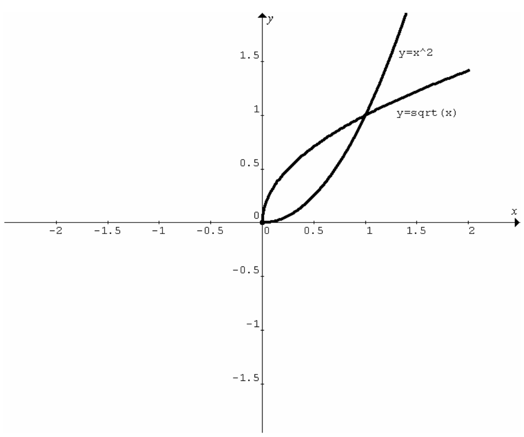
Other examples of inverse pairs are:

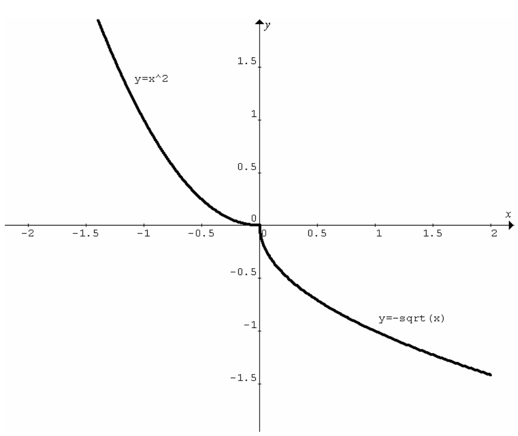
$y = Sin(x)$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y = Sin^{-1}(x)$;

$y = x^2$ for $x \geq 0$ and $y = \sqrt{x}$ ;

$y = x^2$ for $x < 0$ and $y = -\sqrt{x}$ .

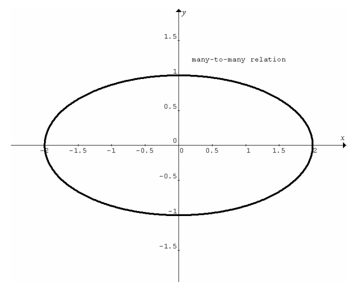


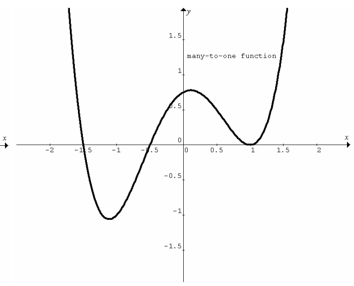



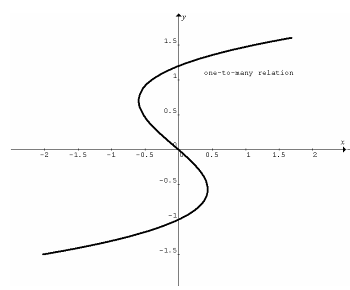



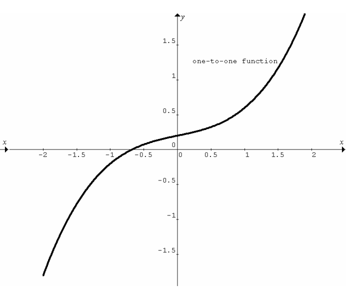

**Relations and functions**

A relation is a set of ordered pairs (points). A function is a relation such that no two points have the same x-coordinate. Use the **vertical line test** to determine whether a relation is a function (cuts through only one point) or not (cuts through more than one point). If it is a function, then it is either a **many-to-one function** or a **one-to-one function**. If it is not a function, then it is either a **many-to-many relation** or a **one-to-many relation**.



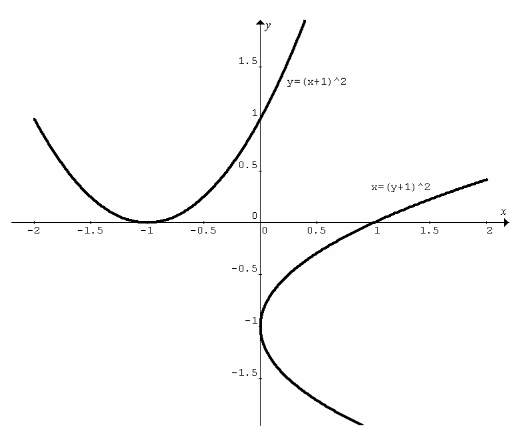




**Inverse functions**

Every relation has an inverse that may or may not be a function. If a relation is a one-to-many relation, then its inverse is a many-to-one function. If a relation is a one-to-one function, then its inverse is also a function (a one-to-one function). If a relation is a many-to-many relation or many-to-one function, then its inverse is not a function.

Use the **horizontal line test** to determine whether the inverse is a function (cuts through only one point) or not (cuts through more than one point).

Example 1 The following two graphs show the original relation $y = (x+1)^2$ that is a many-to-one function, and its inverse $x = (y+1)^2$ that is not a function.



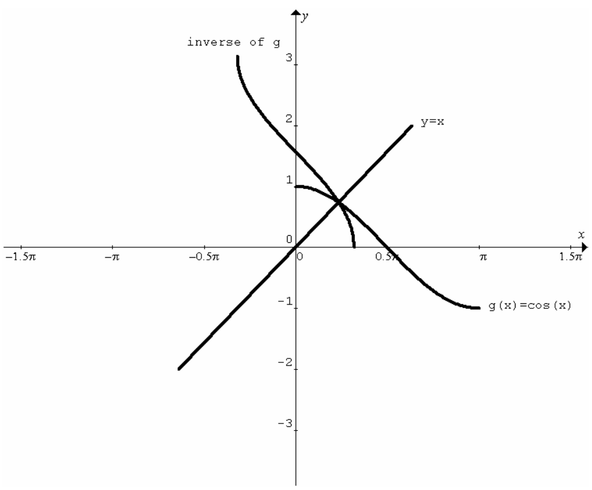
Example 2 The relations $y = Sin(x)$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y = x^2$ for $x \geq 0$ and $y = x^2$ for $x < 0$ are one-to-one functions, $\therefore$ their inverses are also one-to-one functions.

If a relation is a function, **function notations** can be used to represent it, e.g. $y = x^2$ for $x < 0$, $f: R^- \to R, f(x) = x^2$. Since its inverse $y = -\sqrt{x}$ is also a function, use $f^{-1}$ to denote inverse function, $f^{-1}: R^+ \to R, f^{-1}(x) = -\sqrt{x}$.

Example 3 Restrict the domain of $y = \cos x$ to produce a one-to-one function. Use function notations to represent it and its inverse function.

There are an infinite number of possible restrictions of the domain, e.g. $[0, \pi]$. Let the function with this restriction be $g: [0, \pi] \to R, g(x) = \cos x$.

The following graphs show that $g$ and $g^{-1}$ are both one-to-one functions. The graph of $g^{-1}$ is obtained by reflecting $g$ in the line $y = x$.




$g$ has a range of $[-1,1]$, this becomes the domain of $g^{-1}$. The inverse of $y = \cos x$ is $x = \cos y$, the equivalent of $x = \cos y$ is $y = \cos^{-1} x$.

$\therefore g^{-1}: [-1,1] \to R, g^{-1}(x) = \cos^{-1} x$.

Example 4 Given a function with equation $y = \frac{1}{2}\log_e(3x-1) + \frac{5}{2}$, find the domain, range and equation of its inverse with $y$ as the subject.

The given function is defined for $3x - 1 > 0$, i.e. $x > \frac{1}{3}$. $\therefore$ its domain is $\left(\frac{1}{3}, \infty\right)$. Its range is $R$. Hence the domain and range of its inverse are R and $\left(\frac{1}{3}, \infty\right)$ respectively.

The inverse of $y = \frac{1}{2}\log_e(3x-1) + \frac{5}{2}$ is $x = \frac{1}{2}\log_e(3y-1) + \frac{5}{2}$.

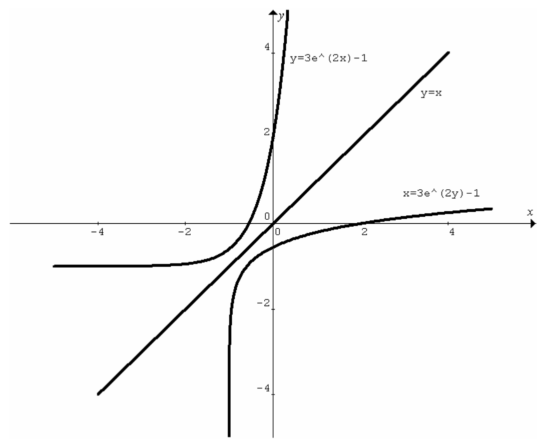
Transpose to make $y$ the subject. $x - \frac{5}{2} = \frac{1}{2}\log_e(3y-1)$,

$2x - 5 = \log_e(3y-1)$, the equivalent is $3y - 1 = e^{2x-5}$,

$3y = e^{2x-5} + 1$, $y = \frac{1}{3}\left(e^{2x-5} + 1\right)$ or $\frac{1}{3}e^{2x-5} + \frac{1}{3}$.

Example 5 Given $f: R \to R, f(x) = 3e^{2x} - 1$. Find $f^{-1}$.

$f$ is a one-to-one function, $\therefore f^{-1}$ exists. The equation for graphing $f$ is $y = 3e^{2x} - 1$. $f$ has $(-1, \infty)$ as its range, $\therefore$ the domain of $f^{-1}$ is $(-1, \infty)$. The graph of $f^{-1}$ is the reflection of $f$ in the line $y = x$.




The equation of $f^{-1}$ is $x = 3e^{2y} - 1$. Transpose to make $y$ the subject.

$x = 3e^{2y} - 1$, $x + 1 = 3e^{2y}$, $\frac{x+1}{3} = e^{2y}$, the equivalent is

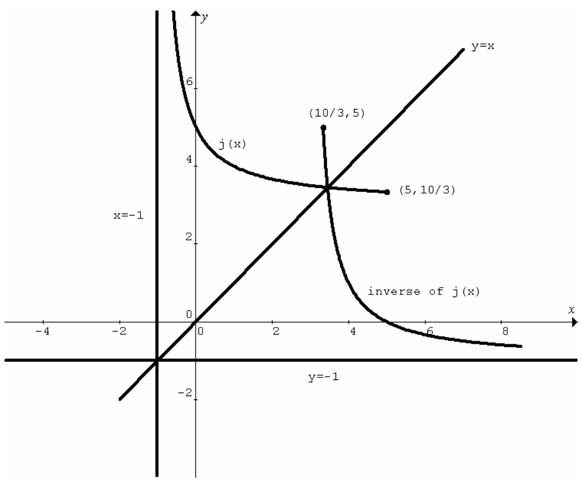
$2y = \log_e\left(\frac{x+1}{3}\right)$, $y = \frac{1}{2}\log_e\left(\frac{x+1}{3}\right)$.

Hence $f^{-1}: (-1, \infty) \to R, f^{-1}(x) = \frac{1}{2}\log_e\left(\frac{x+1}{3}\right)$.

Example 6 Given $j:(-1,5] \to R, j(x)=\frac{2}{x+1}+3$, find the domain, range, asymptote(s) and equation of its inverse.

Consider the given function *j*: Domain $(-1,5]$. Vertical asymptote $x=-1$. Note that horizontal asymptote does not exist because the function has an end point at $x=5$, $y=\frac{2}{5+1}+3=\frac{10}{3}$. Range $\left[\frac{10}{3},\infty\right)$. Equation $y=\frac{2}{x+1}+3$.

The inverse of *j*: Domain $\left[\frac{10}{3},\infty\right)$. Range $(-1,5]$. Horizontal asymptote $y=-1$. Equation $x=\frac{2}{y+1}+3$. Make *y* the subject by transposition. $x-3=\frac{2}{y+1}$, $y+1=\frac{2}{x-3}$, $y=\frac{2}{x-3}-1$.



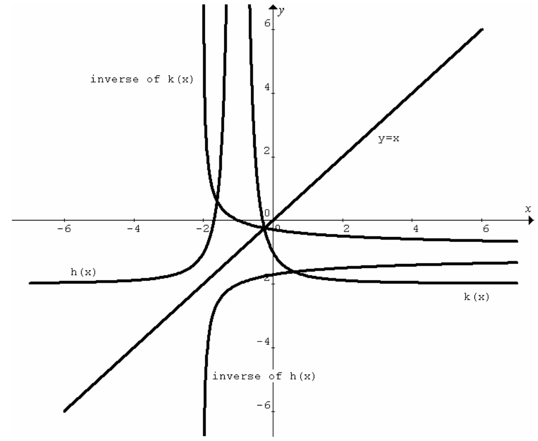

Example 7 Split up $y=\frac{1}{(x+1)^2}-2$ into two one-to-one functions *h* and *k*, and find $h^{-1}$ and $k^{-1}$.

$y=\frac{1}{(x+1)^2}-2$ is a many-to-one function and it is symmetrical about the vertical asymptote $x=-1$. Restrict the function to domain $(-\infty,-1)$ to form one-to-one function *h*. Restrict the function to domain $(-1,\infty)$ to form one-to-one function *k*.

Hence $h:(-\infty,-1)\to R, h(x)=\frac{1}{(x+1)^2}-2$,

and $k:(-1,\infty)\to R, k(x)=\frac{1}{(x+1)^2}-2$.

The graphs of $h^{-1}$ and $k^{-1}$ are obtained by reflecting the graphs of *h* and *k* in the line $y=x$.




Both *h* and *k* have the same horizontal asymptote $y=-2$ and $\therefore$ the same range $(-2,\infty)$. $\therefore$ $h^{-1}$ and $k^{-1}$ have the same vertical asymptote $x=-2$ and domain $(-2,\infty)$. The inverse of $y=\frac{1}{(x+1)^2}-2$ is $x=\frac{1}{(y+1)^2}-2$. The equivalent relation to $x=\frac{1}{(y+1)^2}-2$ is obtained by transposition to make *y* the subject.

$x=\frac{1}{(y+1)^2}-2$, $x+2=\frac{1}{(y+1)^2}$, $(y+1)^2=\frac{1}{x+2}$,

$y+1=\pm\frac{1}{\sqrt{x+2}}$, $y=\pm\frac{1}{\sqrt{x+2}}-1$.

Hence $h^{-1}:(-2,\infty)\to R, h^{-1}(x)=-\frac{1}{\sqrt{x+2}}-1$,

and $k^{-1}:(-2,\infty)\to R, k^{-1}(x)=\frac{1}{\sqrt{x+2}}-1$.

Example 8 Find the domain and equation of the inverse of $y=\frac{1}{9}\left(x-\frac{1}{3}\right)^2+1$, where $x\ge-\frac{8}{3}$.

The given function has an end point $\left(-\frac{8}{3},2\right)$, and a turning point $\left(\frac{1}{3},1\right)$ that is the lowest point of the function. Hence the range is $[1,\infty)$, not $[2,\infty)$. $\therefore$ domain of the inverse is $[1,\infty)$.

Equation of the inverse is $x=\frac{1}{9}\left(y-\frac{1}{3}\right)^2+1$. Make *y* the subject, $x-1=\frac{1}{9}\left(y-\frac{1}{3}\right)^2$, $9(x-1)=\left(y-\frac{1}{3}\right)^2$,

$y-\frac{1}{3}=\pm\sqrt{9(x-1)}$, $y-\frac{1}{3}=\pm3\sqrt{x-1}$, $y=\pm3\sqrt{x-1}+\frac{1}{3}$.

Note: (1) The inverse is not a function. (2) The inverse has an end point $\left(2,-\frac{8}{3}\right)$.

**Inverse functions undo each other**

When a one-to-one function *f* and its inverse function $f^{-1}$ are used to form composite function $f^{-1} \circ f$ or $f \circ f^{-1}$, then $f^{-1} \circ f(x) = x$, $f \circ f^{-1}(x) = x$, i.e. they undo each other.

Example 1 Given $f(x) = (x-2)^3$, find $f^{-1}(x)$. Show that they undo each other.

Equation of *f* is $y = (x-2)^3$, $\therefore$ equation of $f^{-1}$ is $x = (y-2)^3$, i.e. $y = \sqrt[3]{x} + 2$, $\therefore f^{-1}(x) = \sqrt[3]{x} + 2$.

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = \sqrt[3]{f(x)} + 2 = \sqrt[3]{(x-2)^3} + 2 = x$$
$$f \circ f^{-1}(x) = f(f^{-1}(x)) = (f^{-1}(x) - 2)^3 = (\sqrt[3]{x})^3 = x.$$

Example 2 Given $g(x) = \log_e x$, state $g^{-1}(x)$, find $g^{-1} \circ g(x)$ and $g \circ g^{-1}(x)$, and simplify.

$$g^{-1}(x) = e^x.$$
$$g^{-1} \circ g(x) = g^{-1}(g(x)) = e^{g(x)} = e^{\log_e x} = x.$$
$$g \circ g^{-1}(x) = g(g^{-1}(x)) = \log_e e^x = x.$$

Example 3 Simplify $\sqrt{e^{2\log_e(x+1) - \log_e(x-1)^2}}$.

$$\sqrt{e^{2\log_e(x+1) - \log_e(x-1)^2}} = \left(e^{\log_e(x+1)^2 - \log_e(x-1)^2}\right)^{\frac{1}{2}}$$
$$= \left(e^{\log_e \frac{(x+1)^2}{(x-1)^2}}\right)^{\frac{1}{2}} = \left(e^{\log_e \left(\frac{x+1}{x-1}\right)^2}\right)^{\frac{1}{2}} = \left(e^{2\log_e \left(\frac{x+1}{x-1}\right)}\right)^{\frac{1}{2}}$$
$$= e^{\log_e \left(\frac{x+1}{x-1}\right)} = \frac{x+1}{x-1}.$$

**Solving exponential equations algebraically**

Example 1 Solve the following equations.
(a) $4^x = 8$ (b) $10^{2x-3} = 0.01$ (c) $4^{x+1} = 10^{x-1}$.

(a) $4^x = 8$, $(2^2)^x = 2^3$, $2^{2x} = 2^3$, $\therefore 2x = 3$, $x = \frac{3}{2}$.

(b) $10^{2x-3} = 0.01$, $10^{2x-3} = \frac{1}{10^2}$, $10^{2x-3} = 10^{-2}$, $\therefore 2x - 3 = -2$, $x = \frac{1}{2}$.

(c) $4^{x+1} = 10^{x-1}$, 4 and 10 cannot be changed to the same base. Take the log (to base 10 is simpler than to base e in this case) of both sides of the equation. $\log_{10} 4^{x+1} = \log_{10} 10^{x-1}$,
$(x+1)\log_{10} 4 = x - 1$, $(x+1)0.6021 = x - 1$,
$0.6021x + 0.6021 = x - 1$, $x = 4.0259$.

Example 2 Solve $5e^{3x+2} = 6$.

$5e^{3x+2} = 6$, $e^{3x+2} = 1.2$, the equivalent is $3x + 2 = \log_e 1.2$, $3x = \log_e 1.2 - 2$, $\therefore x = \frac{1}{3}(\log_e 1.2 - 2)$ in exact form or $x \approx -0.6059$.

Example 3 Solve $8e^{x+2} = 3e^{2-x}$.

$8e^{x+2} = 3e^{2-x}$, $\frac{e^{x=2}}{e^{2-x}} = \frac{3}{8}$, $e^{x-2-(2-x)} = \frac{3}{8}$, $e^{2x} = \frac{3}{8}$, the equivalent is $2x = \log_e\left(\frac{3}{8}\right)$, $x = \frac{1}{2}\log_e\left(\frac{3}{8}\right)$ in exact form or $x \approx -0.4904$.

Alternative method: $8e^{x+2} = 3e^{2-x}$, $8e^{x+2} - 3e^{2-x} = 0$, $e^{2-x}(8e^{2x} - 3) = 0$. Since $e^{2-x} \neq 0$, $\therefore 8e^{2x} - 3 = 0$, $e^{2x} = \frac{3}{8}$ etc.

Example 4 Solve $3e^{2x} - 4e^x + 1 = 0$.

$3(e^x)^2 - 4(e^x) + 1 = 0$,
$(3e^x - 1)(e^x - 1) = 0$, [factorise by trial and error]
either $3e^x - 1 = 0$ or $e^x - 1 = 0$,
$\therefore e^x = \frac{1}{3}$ or $e^x = 1$, i.e. $x = \log_e\left(\frac{1}{3}\right)$ or $x = 0$.

Example 5 Solve $3e^{2x} + 4e^x + 1 = 0$.

This equation has no real solutions for x, because all three terms are greater than 0, $\therefore$ sum > 0.

Example 6 Solve $e^{2x} - 4e^x - 4 = 0$.

$(e^x)^2 - 4(e^x) - 4 = 0$ cannot be factorised over Q, set of rational numbers, $\therefore$ use the quadratic formula to obtain
$e^x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)}$, $e^x = 2 \pm 2\sqrt{2}$. Since $e^x > 0$, $\therefore e^x = 2 + 2\sqrt{2}$, hence $x = \log_e(2 + 2\sqrt{2})$ in exact form or $x \approx 1.5745$.

Example 7 Find *k* such that the curve $y = 3e^{-kx+2}$ passes through $(-2, 3)$.

When $x = -2$, $y = 3$, $\therefore 3 = 3e^{2k+2}$, $e^{2k+2} = 1$, $2k + 2 = 0$, $\therefore k = -1$.

Example 8 Find $a$ and $b$ such that the curve $y = ae^{bx} + 1$ passes through the points $(-1,2)$ and $(1,4)$.

Use the two points to set up two simultaneous equations:
$(-1,2) \to ae^{-b} + 1 = 2$, $\therefore ae^{-b} = 1$ ………(1)
$(1,4) \to ae^{b} + 1 = 4$, $\therefore ae^{b} = 3$ …………(2)
Eq.(2)/eq.(1), $\dfrac{ae^{b}}{ae^{-b}} = \dfrac{3}{1}$, $e^{2b} = 3$, $2b = \log_e 3$,
$\therefore b = \dfrac{1}{2}\log_e 3 = \log_e \sqrt{3}$ ……(3)
Substitute eq.(3) in eq.(2), $ae^{\log_e \sqrt{3}} = 3$, $a\sqrt{3} = 3$,
$\therefore a = \dfrac{3}{\sqrt{3}} = \sqrt{3}$.

Example 9 Find $b$ and $c$ such that the curve $y = 2e^{bx} + c$ passes through the points $(2,6)$ and $(4,10)$.

$(2,6) \to 2e^{2b} + c = 6$, $\therefore 2e^{2b} = 6 - c$ ………...(1)
$(4,10) \to 2e^{4b} + c = 10$, $\therefore 2e^{4b} = 10 - c$ ……..(2)
Eq.(2)/eq.(1), $\dfrac{2e^{4b}}{2e^{2b}} = \dfrac{10-c}{6-c}$, $\therefore e^{2b} = \dfrac{10-c}{6-c}$ ………(3)
Substitute eq.(3) in eq.(1), $2\left(\dfrac{10-c}{6-c}\right) = 6 - c$,
$\therefore 2(10-c) = (6-c)^2$, $(6-c)^2 - 2(10-c) = 0$,
$(6-c)^2 - 2(6-c) - 8 = 0$.
Factorise to obtain $[(6-c)-4][(6-c)+2] = 0$.
Hence either $6 - c - 4 = 0$, i.e. $c = 2$, or $6 - c + 2 = 0$, i.e. $c = 8$. The second result is not possible because it leads to an impossibility $2e^{2b} = -2$. $\therefore c = 2$ ……(4)
Substitute eq.(4) in eq.(1), $2e^{2b} = 4$, $e^{2b} = 2$, $2b = \log_e 2$,
$b = \dfrac{1}{2}\log_e 2 = \log_e \sqrt{2}$.

**Solving logarithmic equations algebraically**

Example 1 Solve the following equations.
(i) $\log_2 x = 5$ (ii) $\log_2(x^2 - 1) = 3$ (iii) $\log_2\left(\dfrac{1}{x}\right) = 1.5$

(i) $\log_2 x = 5$, the equivalent is $x = 2^5$, $x = 32$.
(ii) $\log_2(x^2 - 1) = 3$, the equivalent is $x^2 - 1 = 2^3$, $x^2 = 9$, $x = \pm 3$.
(iii) $\log_2\left(\dfrac{1}{x}\right) = 1.5$, the equivalent is $\dfrac{1}{x} = 2^{1.5}$, $x = 2^{-1.5}$ or $2^{-\frac{3}{2}}$ or 0.3536

Example 2 Solve $\log_x \sqrt{243} = 2.5$.

$\log_x \sqrt{243} = 2.5$, the equivalent is $\sqrt{243} = x^{2.5}$, $\sqrt{3^5} = x^{2.5}$,
$(3^5)^{\frac{1}{2}} = x^{2.5}$, $3^{2.5} = x^{2.5}$, $\therefore x = 3$

Example 3 Solve $\log_x 5x = 3$, where $x > 0$.

$\log_x 5x = 3$, $\log_x 5 + \log_x x = 3$, $\log_x 5 + 1 = 3$, $\log_x 5 = 2$, the equivalent is $x^2 = 5$, $\therefore x = \sqrt{5}$.

Example 4 Solve $\log_{10} x - \log_{10}(x+1) = 1$.

$\log_{10} x - \log_{10}(x+1) = 1$, $\log_{10}\left(\dfrac{x}{x+1}\right) = 1$, the equivalent is
$\dfrac{x}{x+1} = 10^1$, $\therefore x = 10(x+1)$, $x = 10x + 10$, $9x = -10$, $x = -\dfrac{10}{9}$.

Example 5 Solve $\log_{10}(\sqrt{35} + x) + \log_{10}(\sqrt{35} - x) = 1$.

$\log_{10}(\sqrt{35} + x) + \log_{10}(\sqrt{35} - x) = 1$,
$\log_{10}(\sqrt{35} + x)(\sqrt{35} - x) = 1$,
the equivalent is $(\sqrt{35} + x)(\sqrt{35} - x) = 10^1$,
$\therefore 35 - x^2 = 10$, $x^2 = 25$, $x = \pm 5$.

Example 6 Solve $2\log_e(x-2) - \log_e(x+1) = 0$.

$2\log_e(x-2) - \log_e(x+1) = 0$, $\log_e \dfrac{(x-2)^2}{x+1} = 0$,
$\dfrac{(x-2)^2}{x+1} = 1$, $(x-2)^2 = x + 1$, $x^2 - 4x + 4 = x + 1$,
$x^2 - 5x + 3 = 0$, $\therefore x = \dfrac{5 \pm \sqrt{25 - 12}}{2}$,
i.e. $x = \dfrac{5 + \sqrt{13}}{2}$ or $\dfrac{5 - \sqrt{13}}{2}$. Only the first solution is correct because $x > 2$ for $2\log_e(x-2)$ to be defined.

Example 7 $(-2,1)$ is a point on the curve $y = 2\log_e(1 - ax)$. Find $a$.

$(-2,1) \to 1 = 2\log_e(1 - a(-2))$, $\dfrac{1}{2} = \log_e(1 + 2a)$,
$1 + 2a = e^{\frac{1}{2}}$, $a = \dfrac{1}{2}(\sqrt{e} - 1)$.

Example 8   Find $a$ and $b$ such that $y=\log_e \sqrt{ax+b}+1$ passes through the points $(0,1)$ and $\left(1,\frac{3}{2}\right)$.

$(0,1)\rightarrow 1=\log_e\sqrt{b}+1$, $\therefore \log_e\sqrt{b}=0$, $\sqrt{b}=1$, $\therefore b=1$ ……(1)

$\left(1,\frac{3}{2}\right)\rightarrow \frac{3}{2}=\log_e\sqrt{a+b}+1$, $\log_e\sqrt{a+b}=\frac{1}{2}$, $\sqrt{a+b}=e^{\frac{1}{2}}$,

$\therefore a+b=e$ ……(2)

Substitute eq.(1) in eq.(2), $a=e-1$.

**Algebraic solution of equations involving circular functions**

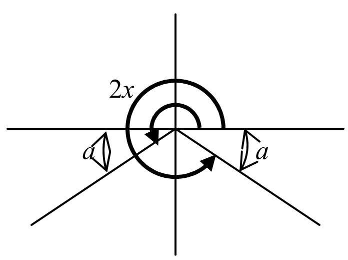
It is prudent to consider the domain when solving an equation involving one or more circular functions.
In this section only equations involving only one of the circular functions are considered.
In solving such equations algebraically, always transpose an equation to make $\sin(kx)$, $\cos(kx)$ or $\tan(kx)$ the subject of the equation.

Example 1   Solve $3\sin(2x)=-2$, where $0<x<6$.

The domain is $0<x<6$, or $0<2x<12$.

$3\sin(2x)=-2$, $\sin(2x)=-\frac{2}{3}$.

Since $\sin(2x)$ has a negative value, $-\frac{2}{3}$, then the 'angle' $2x$ must be in the third or fourth quadrant.




$a=\sin^{-1}\left(\frac{2}{3}\right)=0.7297$ (Calculator)

$2x=\pi+0.7297$, $2\pi-0.7297$, $3\pi+0.7297$, $4\pi-0.7297$

$2x=3.8713, 5.5535, 10.1545, 11.8366$

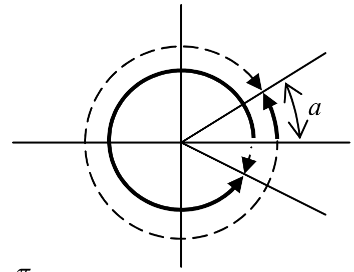
$\therefore x=1.9357, 2.7767, 5.0772, 5.9183$

Example 2   Find the exact solution(s) of $2\sqrt{3}\cos\left(\frac{\theta}{2}\right)-3=0$, where $-4\pi<\theta<4\pi$.

The domain is $-4\pi<\theta<4\pi$, or $-2\pi<\frac{\theta}{2}<2\pi$.

$2\sqrt{3}\cos\left(\frac{\theta}{2}\right)-3=0$, $\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}}{2}$.

Since $\cos\left(\frac{\theta}{2}\right)$ has a positive value, $\frac{\sqrt{3}}{2}$, $\therefore$ the 'angle' $\frac{\theta}{2}$ must be in the first or fourth quadrant.




$a=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

$\frac{\theta}{2}=-2\pi+\frac{\pi}{6}$, $-\frac{\pi}{6}$, $\frac{\pi}{6}$, $2\pi-\frac{\pi}{6}$

$\therefore \frac{\theta}{2}=-\frac{11\pi}{6}$, $-\frac{\pi}{6}$, $\frac{\pi}{6}$, $\frac{11\pi}{6}$   $\therefore \theta=-\frac{11\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{11\pi}{3}$.

Example 3   Find the exact coordinates of the intersections of $y=3\sin(2x)+1$ and $y=\sqrt{3}\cos(2x)+1$, where $-\pi<x<\pi$.

Solve the two equations simultaneously to find the intersections.

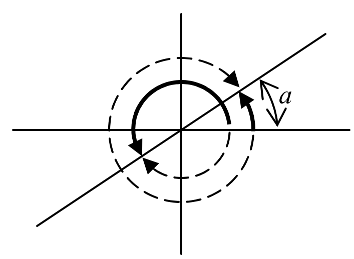
$y=3\sin(2x)+1$ ………(1)   $y=\sqrt{3}\cos(2x)+1$ ……(2)

The domain is $-\pi<x<\pi$, or $-2\pi<2x<2\pi$.

Substitute eq.(1) in eq.(2), $3\sin(2x)+1=\sqrt{3}\cos(2x)+1$,

$\therefore 3\sin(2x)=\sqrt{3}\cos(2x)$, $\frac{\sin(2x)}{\cos(2x)}=\frac{\sqrt{3}}{3}$, $\tan(2x)=\frac{1}{\sqrt{3}}$.

Since $\tan(2x)$ has a positive value, $\frac{1}{\sqrt{3}}$, the 'angle' $2x$ must be in the first or third quadrant.




$a=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$2x=-2\pi+\frac{\pi}{6}$, $-\pi+\frac{\pi}{6}$, $\frac{\pi}{6}$, $\pi+\frac{\pi}{6}$,

$\therefore 2x=-\frac{11\pi}{6}$, $-\frac{5\pi}{6}$, $\frac{\pi}{6}$, $\frac{7\pi}{6}$, $\therefore x=-\frac{11\pi}{12}$, $-\frac{5\pi}{12}$, $\frac{\pi}{12}$, $\frac{7\pi}{12}$.

Substitute each of these $(2x)$ values in eq.(1),

$y=3\sin\left(-\frac{11\pi}{6}\right)+1=3\left(\frac{1}{2}\right)+1=\frac{5}{2}$,

$y=3\sin\left(-\frac{5\pi}{6}\right)+1=3\left(-\frac{1}{2}\right)+1=-\frac{1}{2}$,

$y=3\sin\left(\frac{\pi}{6}\right)+1=3\left(\frac{1}{2}\right)+1=\frac{5}{2}$,

$y=3\sin\left(\frac{7\pi}{6}\right)+1=3\left(-\frac{1}{2}\right)+1=-\frac{1}{2}$. The coordinates are $\left(-\frac{11\pi}{12},\frac{5}{2}\right)$, $\left(-\frac{5\pi}{12},-\frac{1}{2}\right)$, $\left(\frac{\pi}{12},\frac{5}{2}\right)$ and $\left(\frac{7\pi}{12},-\frac{1}{2}\right)$.

Example 4 Solve $\sin^2(x) = \sin(x)\cos(x)$ for $x$, where $0 \le x \le \frac{\pi}{2}$. Note: $\sin^2(x)$ is the proper notation for $(\sin(x))^2$.

Caution: Do not divide both sides of the equation by $\sin(x)$, because $\sin(x) = 0$ is a possibility. See below.

$\sin^2(x) = \sin(x)\cos(x)$, $\sin^2(x) - \sin(x)\cos(x) = 0$, common factor, $\sin(x)(\sin(x) - \cos(x)) = 0$.

$\therefore$ either $\sin(x) = 0$, $\therefore x = 0$,

or $\sin(x) - \cos(x) = 0$, $\frac{\sin(x) - \cos(x)}{\cos(x)} = \frac{0}{\cos(x)}$ for $\cos(x) \ne 0$,

$\therefore \tan(x) - 1 = 0$, $\tan(x) = 1$, $x = \frac{\pi}{4}$.

The two possible solutions for $x$ are 0, $\frac{\pi}{4}$.

Example 5 Find $t$ when $x = 10\sin\left(\frac{\pi t}{12}\right) - 12$ is a minimum, where $0 < t < 48$.

The domain is $0 < t < 48$, or $0 < \frac{\pi t}{12} < 4\pi$.

Minimum of $x$ is $-22$, it occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$,

$\therefore \frac{\pi t}{12} = \frac{3\pi}{2}$, $\frac{7\pi}{2}$. $\therefore t = 18$, 42.