

Further Calculus Part I

Review of basic properties and rules of differentiation

Properties of differentiation:

$$1) \frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$$

$$2) \frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

Example 1 Find the derivative of $\frac{\sqrt{3}}{2\pi} (2e^{\pi x} - \sqrt{3}x^\pi)$.

$$\begin{aligned} \frac{d}{dx} \frac{\sqrt{3}}{2\pi} (2e^{\pi x} - \sqrt{3}x^\pi) &= \frac{\sqrt{3}}{2\pi} \frac{d}{dx} (2e^{\pi x} - \sqrt{3}x^\pi) \\ &= \frac{\sqrt{3}}{2\pi} \left(2 \frac{d}{dx} e^{\pi x} - \sqrt{3} \frac{d}{dx} x^\pi \right) \\ &= \frac{\sqrt{3}}{2\pi} (2\pi e^{\pi x} - \sqrt{3}\pi x^{\pi-1}) \\ &= \sqrt{3}e^{\pi x} - \frac{3}{2}x^{\pi-1} \end{aligned}$$

Example 2 Given $f'(x) = \cos^2 x$ and $y = 2\left(f(x) - \frac{x}{2}\right)$,

evaluate $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$.

$$\begin{aligned} y &= 2\left(f(x) - \frac{x}{2}\right), \quad \frac{dy}{dx} = 2\left(f'(x) - \frac{1}{2}\right) = 2f'(x) - 1 \\ &= 2\cos^2 x - 1 = \cos 2x = \cos \pi = -1. \end{aligned}$$

Rules of differentiation:

The **product rule** is used to differentiate functions that are the products of two other functions.

If $y = u(x)v(x)$, then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$.

Alternative notations: $y' = vu' + uv'$.

Example 1 Differentiate $2e^{\frac{x}{2}} \sin\left(2x - \frac{\pi}{2}\right)$.

$$\begin{aligned} \frac{d}{dx} 2e^{\frac{x}{2}} \sin\left(2x - \frac{\pi}{2}\right) &= 2e^{\frac{x}{2}} \frac{d}{dx} \sin\left(2x - \frac{\pi}{2}\right) + \sin\left(2x - \frac{\pi}{2}\right) \frac{d}{dx} 2e^{\frac{x}{2}} \\ &= 4e^{\frac{x}{2}} \cos\left(2x - \frac{\pi}{2}\right) + e^{\frac{x}{2}} \sin\left(2x - \frac{\pi}{2}\right) \\ &= e^{\frac{x}{2}} \left[4 \cos\left(2x - \frac{\pi}{2}\right) + \sin\left(2x - \frac{\pi}{2}\right) \right]. \end{aligned}$$

Example 2 Find $f'(x)$, given $f(x) = \sqrt{2x+1}(x^2 + x - 1)$. Hence evaluate $f'(0)$.

$$\begin{aligned} f(x) &= \sqrt{2x+1}(x^2 + x - 1), \\ f'(x) &= \sqrt{2x+1} \frac{d}{dx}(x^2 + x - 1) + (x^2 + x - 1) \frac{d}{dx} \sqrt{2x+1} \\ &= \sqrt{2x+1}(2x+1) + (x^2 + x - 1) \frac{1}{\sqrt{2x+1}} \\ &= \sqrt{2x+1}(2x+1) + (x^2 + x - 1) \frac{\sqrt{2x+1}}{2x+1} \\ &= \sqrt{2x+1} \left(\frac{(2x+1)^2 + (x^2 + x - 1)}{2x+1} \right) \\ &= \sqrt{2x+1} \left(\frac{5x^2 + 5x}{2x+1} \right) = \frac{5x(x+1)\sqrt{2x+1}}{2x+1}. \end{aligned}$$

Hence $f'(0) = 0$.

The **quotient rule** is used to differentiate expressions of the form $\frac{u(x)}{v(x)}$.

If $y = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$.

Alternative notations: $y' = \frac{vu' - uv'}{v^2}$.

Example 1 Differentiate $\frac{2 \log_e bx}{x^2}$.

$$\begin{aligned} \frac{d}{dx} \left(\frac{2 \log_e bx}{x^2} \right) &= \frac{x^2 \frac{d}{dx}(2 \log_e bx) - (2 \log_e bx) \frac{d}{dx} x^2}{(x^2)^2} \\ &= \frac{x^2 \left(\frac{2}{x} \right) - (2 \log_e bx)(2x)}{x^4} \\ &= \frac{2x(1 - 2 \log_e bx)}{x^4} = \frac{2(1 - 2 \log_e bx)}{x^3}. \end{aligned}$$

Example 2 Find an equation of the tangent line to the curve $y = \frac{x}{1+x^2}$ at $x = 2$.

Gradient function: $\frac{dy}{dx} = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$.

At $x = 2$, $m_T = \frac{1-2^2}{(1+2^2)^2} = -0.12$ and $y = 0.4$.

Equation of the tangent: $y - y_1 = m_T(x - x_1)$,

$$y - 0.4 = -0.12(x - 2),$$

$$y = -0.12x + 0.28.$$

The chain rule is used to find the derivatives of composite functions. If $y = f(u(x))$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Alternative notations: $y' = f'(u)u'$.

Example 1 Differentiate $\sqrt{\sin\left(x - \frac{\pi}{3}\right)}$.

Let $y = \sqrt{u}$, where $u = \sin\left(x - \frac{\pi}{3}\right)$.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\sin\left(x - \frac{\pi}{3}\right)}} \text{ and } \frac{du}{dx} = \cos\left(x - \frac{\pi}{3}\right).$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\cos\left(x - \frac{\pi}{3}\right)}{2\sqrt{\sin\left(x - \frac{\pi}{3}\right)}}.$$

Example 2 Find the derivative of $\sin\sqrt{x - \frac{\pi}{3}}$.

Let $y = \sin u$, where $u = \sqrt{x - \frac{\pi}{3}}$.

$$\frac{dy}{du} = \cos u = \cos\sqrt{x - \frac{\pi}{3}} \text{ and } \frac{du}{dx} = \frac{1}{2\sqrt{x - \frac{\pi}{3}}}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\cos\sqrt{x - \frac{\pi}{3}}}{2\sqrt{x - \frac{\pi}{3}}}.$$

Derivatives of inverse trigonometric (circular) functions

Inverse sine functions:

$f(x)$	$f'(x)$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\sin^{-1}(ax)$	$\frac{a}{\sqrt{1-(ax)^2}}$
$\sin^{-1}\left(\frac{x}{a}\right), a > 0$	$\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1}\left(\frac{x}{a}\right), a < 0$	$-\frac{1}{\sqrt{a^2-x^2}}$

Example 1 Differentiate $\frac{1}{2}\sin^{-1}(-2x)$.

$$\text{Let } f(x) = \frac{1}{2}\sin^{-1}(-2x)$$

$$f'(x) = \frac{1}{2} \frac{-2}{\sqrt{1-(-2x)^2}} = -\frac{1}{\sqrt{1-4x^2}}.$$

Example 2 Find $\frac{d}{dx}\left(2\sin^{-1}\frac{x}{3} + \frac{\pi}{6}\right)$.

$$\frac{d}{dx}\left(2\sin^{-1}\frac{x}{3} + \frac{\pi}{6}\right) = 2\frac{d}{dx}\sin^{-1}\frac{x}{3} = \frac{2}{\sqrt{9-x^2}}.$$

Example 3 Find the derivative of $\sin^{-1}\left(\frac{2}{x}\right)$.

Let $y = \sin^{-1}u$, where $u = \frac{2}{x}$.

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}} = \frac{x}{\sqrt{x^2-4}} \text{ and } \frac{du}{dx} = -\frac{2}{x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{2}{x\sqrt{x^2-4}}.$$

Inverse cosine functions:

$f(x)$	$f'(x)$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(ax)$	$-\frac{a}{\sqrt{1-(ax)^2}}$
$\cos^{-1}\left(\frac{x}{a}\right), a > 0$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right), a < 0$	$\frac{1}{\sqrt{a^2-x^2}}$

Example 1

Show that the derivative of $\cos^{-1}\left(\frac{ax}{b}\right)$ is $-\frac{a}{\sqrt{b^2-(ax)^2}}$

for $a, b > 0$. Hence differentiate $\cos^{-1}\left(\frac{2x}{3}\right)$.

Let $y = \cos^{-1}(u)$, where $u = \frac{ax}{b}$.

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}} = -\frac{1}{\sqrt{1-\left(\frac{ax}{b}\right)^2}} = -\frac{b}{\sqrt{b^2-(ax)^2}} \text{ and}$$

$$\frac{du}{dx} = \frac{a}{b}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{a}{\sqrt{b^2-(ax)^2}}.$$

$$\text{Hence } \frac{d}{dx}\cos^{-1}\left(\frac{2x}{3}\right) = -\frac{2}{\sqrt{9-4x^2}}.$$

Inverse tangent functions:

$f(x)$	$f'(x)$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan^{-1}(ax)$	$\frac{a}{1+(ax)^2}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$

Example 1 Show that the derivative of $\tan^{-1}\left(\frac{ax}{b}\right)$ is $\frac{ab}{b^2+(ax)^2}$. Hence differentiate $\tan^{-1}\left(\frac{-3x}{2}\right)$.

Let $y = \tan^{-1}(u)$, where $u = \frac{ax}{b}$.

$$\frac{dy}{du} = \frac{1}{1+u^2} = \frac{1}{1+\left(\frac{ax}{b}\right)^2} = \frac{b^2}{b^2+(ax)^2} \text{ and } \frac{du}{dx} = \frac{a}{b}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{ab}{b^2+(ax)^2}.$$

$$\text{Hence } \frac{d}{dx} \tan^{-1}\left(\frac{-3x}{2}\right) = -\frac{6}{4+9x^2}.$$

Second derivatives

The derivative of the derivative of a function is called the second derivative of the function.

If $y = f(x)$, the second derivative is denoted by

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Example 1 Find $\frac{d^2y}{dx^2}$, given $y = \log_e b(x+1)$.

$$y = \log_e b(x+1), \frac{dy}{dx} = \frac{1}{x+1}, \frac{d^2y}{dx^2} = -\frac{1}{(x+1)^2}.$$

Example 2 Show that $\frac{d^2y}{dx^2} = 4y$ for $y = 3e^{-2x}$.

$$y = 3e^{-2x}, \frac{dy}{dx} = -6e^{-2x},$$

$$\frac{d^2y}{dx^2} = 12e^{-2x} = 4 \times 3e^{-2x} = 4y.$$

Example 3 The position of an object moving along a straight path is given by $x = 5 - 10\cos^2 t$ for $t \geq 0$. Find the position, velocity and acceleration of the object at $t = \frac{\pi}{2}$.

$$x = 5 - 10\cos^2 t = -5(2\cos^2 t - 1) = -5\cos 2t,$$

$$v = \frac{dx}{dt} = 10\sin 2t, a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 20\cos 2t.$$

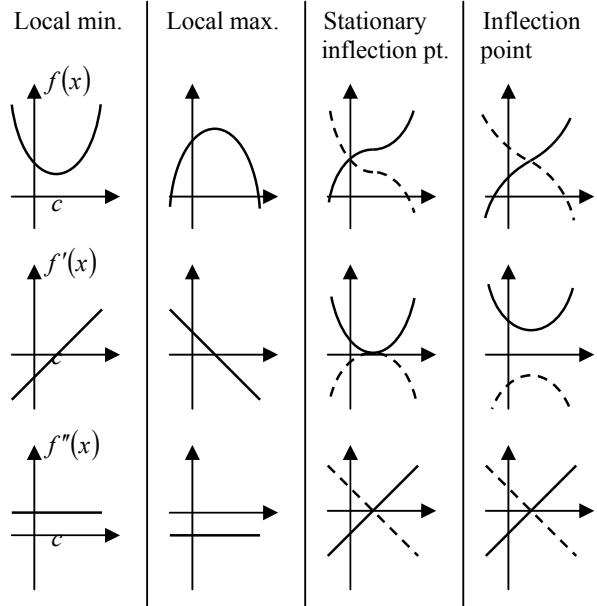
$$\text{At } t = \frac{\pi}{2}, x = -5\cos \pi = 5,$$

$$v = 10\sin \pi = 0, a = 20\cos \pi = -20.$$

Application of second derivative to the analysis of graphs of functions

Given a continuous and ‘smooth’ function $f(x)$,

$f'(c)$	$f''(c)$	at $x = c$
= 0	> 0	local minimum
= 0	< 0	local maximum
= 0	= 0	stationary inflection point
≠ 0	= 0	inflection point



Example 1 Find the stationary points and inflection points of the function $e^x \sin x, 0 \leq x \leq 2\pi$. Sketch the graph.

$$\text{Let } y = e^x \sin x, \frac{dy}{dx} = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x),$$

$$\frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + e^x(\cos x + \sin x) = 2e^x \cos x.$$

$$\text{Inflection points: } \frac{d^2y}{dx^2} = 0, \therefore 2e^x \cos x = 0, \therefore \cos x = 0,$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$\therefore y = e^{\frac{\pi}{2}}, -e^{\frac{3\pi}{2}}. \quad \left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right), \left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right).$$

$$\text{Stationary points: } \frac{dy}{dx} = 0, \therefore e^x(\cos x + \sin x) = 0,$$

$$\therefore \cos x + \sin x = 0, 1 + \tan x = 0, \tan x = -1, \\ \therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

$$\therefore y = e^{\frac{3\pi}{4}} \sin \frac{3\pi}{4} = \frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2} \text{ or } e^{\frac{7\pi}{4}} \sin \frac{7\pi}{4} = -\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2}.$$

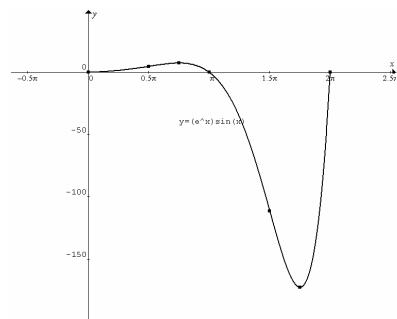
$$\left(\frac{3\pi}{4}, \frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2} \right), \left(\frac{7\pi}{4}, -\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2} \right).$$

$$\text{At } x = \frac{3\pi}{4}, \frac{d^2y}{dx^2} = 2e^x \cos x = 2e^{\frac{3\pi}{4}} \cos \frac{3\pi}{4} < 0,$$

$$\therefore \left(\frac{3\pi}{4}, \frac{\sqrt{2}e^{\frac{3\pi}{4}}}{2} \right) \text{ is a local maximum.}$$

$$\text{At } x = \frac{7\pi}{4}, \frac{d^2y}{dx^2} = 2e^x \cos x = 2e^{\frac{7\pi}{4}} \cos \frac{7\pi}{4} > 0,$$

$$\therefore \left(\frac{7\pi}{4}, -\frac{\sqrt{2}e^{\frac{7\pi}{4}}}{2} \right) \text{ is a local minimum.}$$



Example 2 Find and determine the nature of the stationary points and inflection points of the function with equation

$$y = \frac{x}{x^2 + 1}. \text{ Does it have an asymptote? Sketch its graph.}$$

$$y = \frac{x}{x^2 + 1}, y' = \frac{vu' - uv'}{v^2} = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2},$$

$$y'' = \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2(x^2 + 1)2x)}{(x^2 + 1)^4}$$

$$= \frac{2x(x^2 + 1)(x^2 - 3)}{(x^2 + 1)^4}.$$

$$\text{Stationary points: } y' = 0, \frac{1 - x^2}{(x^2 + 1)^2} = 0, \therefore 1 - x^2 = 0,$$

$$\therefore x = -1, 1, \therefore y = -\frac{1}{2}, \frac{1}{2}. \quad \left(-1, -\frac{1}{2} \right), \left(1, \frac{1}{2} \right).$$

At $x = -1, y'' > 0, \therefore \left(-1, -\frac{1}{2} \right)$ is a local minimum.

At $x = 1, y'' < 0, \therefore \left(1, \frac{1}{2} \right)$ is a local maximum.

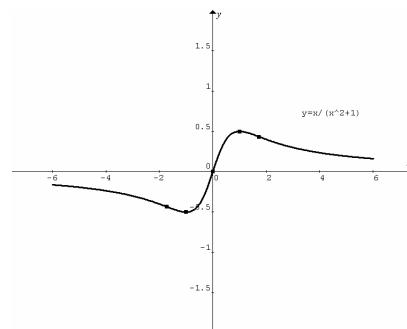
Inflection points: $y'' = 0,$

$$\frac{2x(x^2 + 1)(x^2 - 3)}{(x^2 + 1)^4} = 0,$$

$$\therefore 2x(x^2 + 1)(x^2 - 3) = 0,$$

$$\therefore x = 0, \pm\sqrt{3}, \therefore y = 0, \pm\frac{\sqrt{3}}{4}.$$

$$(0,0), \left(-\sqrt{3}, -\frac{\sqrt{3}}{4} \right), \left(\sqrt{3}, \frac{\sqrt{3}}{4} \right).$$



Horizontal asymptote: $y = 0$.

Implicit differentiation

This topic is an application of the chain rule. The derivatives of relations like $x^2 + y^2 = 9$ and $3xy^2 = x + y$ can be obtained by implicit differentiation. This method eliminates the tedious task of rewriting a relation to make y the subject.

Example 1 Given $x^2 + y^2 = 9$, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x .

Differentiate both sides of $x^2 + y^2 = 9$ with respect to x , use the chain rule if necessary. Note: y is a function of x .

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(9),$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0,$$

$$2x + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 0,$$

$$\therefore 2x + 2y \frac{dy}{dx} = 0, \frac{dy}{dx} = -\frac{x}{y}.$$

$$\text{Since } y = \pm\sqrt{9 - x^2}, \therefore \frac{dy}{dx} = \pm\frac{x}{\sqrt{9 - x^2}}.$$

Example 2 Given $3xy^2 = x + y$, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y . Hence evaluate $\frac{dy}{dx}$ at $x = 2$.

Differentiate both sides of $3xy^2 = x + y$ with respect to x ,

$$\frac{d}{dx}(3xy^2) = \frac{d}{dx}(x + y),$$

$$y^2 \frac{d}{dx}(3x) + 3x \frac{d}{dx}(y^2) = \frac{d}{dx}(x) + \frac{d}{dx}(y),$$

$$\therefore 3y^2 + 3x \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 1 + \frac{dy}{dx}, \therefore 3y^2 + 6xy \frac{dy}{dx} = 1 + \frac{dy}{dx},$$

$$6xy \frac{dy}{dx} - \frac{dy}{dx} = 1 - 3y^2, (6xy - 1) \frac{dy}{dx} = 1 - 3y^2,$$

$$\therefore \frac{dy}{dx} = \frac{1 - 3y^2}{6xy - 1}.$$

$$\text{At } x = 2, 6y^2 = 2 + y, 6y^2 - y - 2 = 0,$$

$$(2y+1)(3y-2) = 0, \therefore y = -\frac{1}{2}, \frac{2}{3}.$$

$$\text{Hence } \frac{dy}{dx} = \frac{1}{28}, -\frac{1}{21}.$$

Related rates

In a **related rates** problem, the rate of change of one quantity is expressed in terms of the rate of change of another quantity. This requires one to find a relationship (i.e. an equation) between the two quantities, and using the chain rule to differentiate both sides with respect to time or other variables.

Example 1 Air is pumped into a spherical balloon at a constant rate of $80 \text{ cm}^3 \text{s}^{-1}$. How fast does the radius increase when it is 30 cm?

The two quantities in this problem are volume $V \text{ cm}^3$ and radius $r \text{ cm}$. Their relationship is $V = \frac{4}{3}\pi r^3$.

Differentiate both sides with respect to time t :

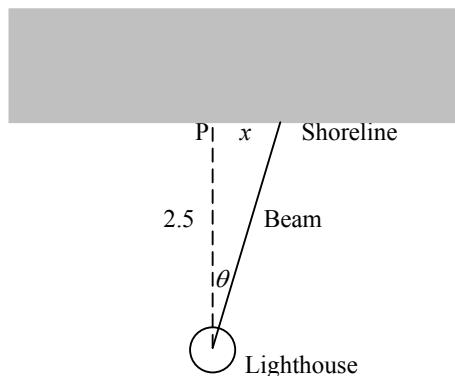
$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt},$$

$$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}.$$

$$\text{Given } \frac{dV}{dt} = +80, \text{ at } r = 30,$$

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2} \times 80 = \frac{1}{45\pi} \text{ cms}^{-1}.$$

Example 2 A lighthouse is on a small island 2.5 km from the nearest point P on a straight shoreline. The light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 0.5 km from P?



Let x km be the distance of the beam from P along the shoreline, and θ the angle in radians between the beam and the dotted line.

The two quantities in this problem are x and θ . Their relationship is $x = 2.5 \tan \theta$.

Differentiate both sides with respect to time t (hours):

$$\frac{dx}{dt} = \frac{d}{dt}(2.5 \tan \theta) = \frac{d}{d\theta}(2.5 \tan \theta) \frac{d\theta}{dt} = 2.5 \sec^2 \theta \frac{d\theta}{dt}.$$

$$\text{Given } \frac{d\theta}{dt} = 4 \text{ rev min}^{-1} = 8\pi \text{ radians min}^{-1}$$

$$= 480\pi \text{ radians hour}^{-1}, \text{ and at } x = 0.5,$$

$$\theta = \tan^{-1}\left(\frac{0.5}{2.5}\right) = 0.1974.$$

$$\therefore \frac{dx}{dt} = 2.5(\sec^2 0.1974) \times 480\pi = 3921 \text{ kmh}^{-1}.$$

Alternative method:

$$\text{Express the relationship as } \theta = \tan^{-1}\left(\frac{x}{2.5}\right).$$

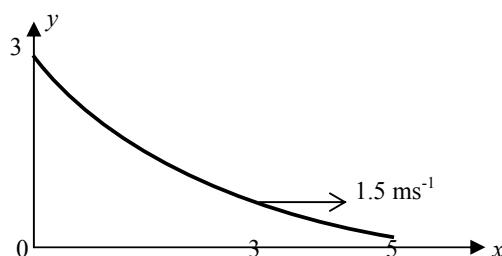
Differentiate both sides with respect to time t (hours):

$$\frac{d\theta}{dt} = \frac{d}{dt} \tan^{-1}\left(\frac{x}{2.5}\right) = \frac{d}{dx} \tan^{-1}\left(\frac{x}{2.5}\right) \frac{dx}{dt} = \frac{2.5}{2.5^2 + x^2} \frac{dx}{dt},$$

$$\therefore \frac{dx}{dt} = \frac{2.5^2 + x^2}{2.5} \frac{d\theta}{dt} = \frac{2.5^2 + 0.5^2}{2.5} \times 480\pi = 3921 \text{ kmh}^{-1}.$$

Example 3 A playground slide has a profile given by

$y = 3e^{-\frac{x}{2}}$, $x \in [0, 5]$. A kid slides down the slope and has a horizontal speed of 1.5 ms^{-1} at $x = 3 \text{ m}$. Calculate the vertical speed at that instant.



The two quantities in this problem are x and y . Their relationship is $y = 3e^{-\frac{x}{2}}$.

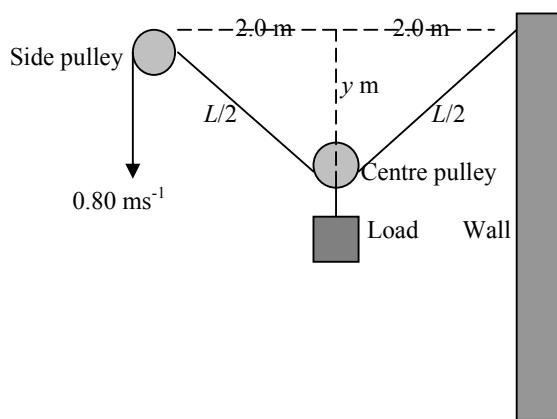
Differentiate both sides with respect to time t (s):

$$\frac{dy}{dt} = \frac{d}{dt} 3e^{-\frac{x}{2}} = \frac{d}{dx} 3e^{-\frac{x}{2}} \frac{dx}{dt} = -\frac{3}{2} e^{-\frac{x}{2}} \frac{dx}{dt}.$$

Given $\frac{dx}{dt} = 1.5$ at $x = 3$,

$\therefore \frac{dy}{dt} = -\frac{3}{2} e^{-\frac{3}{2}} \times 1.5 = -0.50 \text{ ms}^{-1}$, i.e. the vertical speed of the kid is 0.50 ms^{-1} . The negative sign indicates downward motion.

Example 4 Consider the following pulley system in lifting a load. The side pulley is 4.0 m from the wall. The rope is pulled at a constant speed of 0.80 ms^{-1} . How fast does the load rise when the centre pulley is 1.0 m lower than the side pulley?



Let $L \text{ m}$ be the length of the rope between the side pulley and the wall when the centre pulley is $y \text{ m}$ lower than the side pulley.

$$\left(\frac{L}{2}\right)^2 = 2^2 + y^2, \therefore y = \sqrt{\frac{L^2}{4} - 4}.$$

Differentiate both sides with respect to time t (s):

$$\frac{dy}{dt} = \frac{d}{dt} \sqrt{\frac{L^2}{4} - 4} = \frac{d}{dL} \sqrt{\frac{L^2}{4} - 4} \frac{dL}{dt} = \frac{L}{4\sqrt{\frac{L^2}{4} - 4}} \frac{dL}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{L}{2\sqrt{L^2 - 16}} \frac{dL}{dt}.$$

Given $\frac{dL}{dt} = -0.80$, when $y = 1.0$, $L = 2\sqrt{5}$,

$\therefore \frac{dy}{dt} = -0.89$, i.e. y decreases at 0.89 ms^{-1} and the load rises at 0.89 ms^{-1} .

Review of some basic anti-differentiation properties

$$1) \int f'(x)dx = f(x) + C$$

$$2) \int kh(x)dx = k \int h(x)dx$$

$$3) \int (p(x) \pm q(x))dx = \int p(x)dx \pm \int q(x)dx$$

Example 1 Find $\int 2\left(e^{2x} - \frac{1}{x}\right)dx$.

$$\begin{aligned} \int 2\left(e^{2x} - \frac{1}{x}\right)dx &= 2 \int \left(e^{2x} - \frac{1}{x}\right)dx \\ &= 2 \left(\int e^{2x} dx - \int \frac{1}{x} dx \right) = 2 \left(\frac{1}{2} e^{2x} - \log_e|x| + C \right) \\ &= e^{2x} - 2 \log_e|x| + C \end{aligned}$$

Example 2 Anti-differentiate $\frac{5x^2 - 2x + x^{-1}}{x}$.

$$\begin{aligned} \int \frac{5x^2 - 2x + x^{-1}}{x} dx &= \int (5x - 2 + x^{-2}) dx \\ &= \int 5xdx - \int 2dx + \int x^{-2} dx \\ &= \frac{5}{2}x^2 - 2x - \frac{1}{x} + C \end{aligned}$$

Example 3 If $f'(x) = \cos 2x - \sin 2x$, find $f(x)$, given $f\left(\frac{\pi}{2}\right) = 0$.

$$\begin{aligned} f'(x) &= \cos 2x - \sin 2x, \therefore f(x) = \int (\cos 2x - \sin 2x) dx \\ &= \int \cos 2xdx - \int \sin 2xdx = \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + C. \\ \text{Given } f\left(\frac{\pi}{2}\right) &= 0, \therefore \frac{1}{2} \sin 2\left(\frac{\pi}{2}\right) + \frac{1}{2} \cos 2\left(\frac{\pi}{2}\right) + C = 0, \\ \therefore C &= \frac{1}{2}. \text{ Hence } f(x) = \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + \frac{1}{2}. \end{aligned}$$

Example 4 Differentiate $\log_e(\sin(2x-1))$ and hence find an anti-derivative of $\cot(2x-1)$.

Apply the chain rule: Let $y = \log_e u$, where $u = \sin(2x-1)$,

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin(2x-1)}, \frac{du}{dx} = 2\cos(2x-1).$$

$$\therefore \frac{d}{dx} \log_e(\sin(2x-1)) = \frac{dy}{du} \times \frac{du}{dx} = \frac{2\cos(2x-1)}{\sin(2x-1)} = 2\cot(2x-1)$$

$$\therefore \cot(2x-1) = \frac{1}{2} \frac{d}{dx} \log_e(\sin(2x-1)).$$

$$\begin{aligned} \text{Hence } \int \cot(2x-1)dx &= \int \frac{1}{2} \frac{d}{dx} \log_e(\sin(2x-1))dx \\ &= \frac{1}{2} \int \frac{d}{dx} \log_e(\sin(2x-1))dx = \frac{1}{2} \log_e(\sin(2x-1)). \end{aligned}$$

Anti-differentiation techniques

Anti-differentiation of $\frac{1}{\sqrt{a^2 - x^2}}$ **and** $\frac{a}{a^2 + x^2}$

$f(x)$	$\int f(x)dx$
$\frac{1}{\sqrt{a^2 - x^2}}, a > 0$	$\text{Sin}^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{1 - (ax)^2}}, a > 0$	$\frac{1}{a} \text{Sin}^{-1}(ax) + C$
$\frac{-1}{\sqrt{a^2 - x^2}}, a > 0$	$\text{Cos}^{-1}\left(\frac{x}{a}\right) + C$
$\frac{-1}{\sqrt{1 - (ax)^2}}, a > 0$	$\frac{1}{a} \text{Cos}^{-1}(ax) + C$
$\frac{a}{a^2 + x^2}, a > 0$	$\text{Tan}^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{1 + (ax)^2}, a > 0$	$\frac{1}{a} \text{Tan}^{-1}(ax) + C$

Example 1 Find $\int \frac{-1}{\sqrt{2-x^2}} dx$.

$$\int \frac{-1}{\sqrt{2-x^2}} dx = \int \frac{-1}{\sqrt{(\sqrt{2})^2 - x^2}} dx = \text{Cos}^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Example 2 Find an anti-derivative of $\frac{3}{5+x^2}$.

$$\begin{aligned} \int \frac{3}{5+x^2} dx &= \int \frac{\frac{3}{\sqrt{5}} \times \sqrt{5}}{(\sqrt{5})^2 + x^2} dx = \frac{3}{\sqrt{5}} \int \frac{\sqrt{5}}{(\sqrt{5})^2 + x^2} dx \\ &= \frac{3}{\sqrt{5}} \text{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) \text{ or } \frac{3\sqrt{5}}{5} \text{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right). \end{aligned}$$

Example 3 Anti-differentiate $\frac{1}{\sqrt{1-4x^2}}$.

$$\begin{aligned} \int \frac{1}{\sqrt{1-4x^2}} dx &= \int \frac{1}{\sqrt{4\left(\frac{1}{4}-x^2\right)}} dx = \int \frac{1}{2\sqrt{\left(\frac{1}{2}\right)^2 - x^2}} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}} dx = \frac{1}{2} \text{Sin}^{-1}\left(\frac{x}{\frac{1}{2}}\right) + C = \frac{1}{2} \text{Sin}^{-1}(2x) + C. \end{aligned}$$

Alternative method:

$$\int \frac{1}{\sqrt{1-4x^2}} dx = \int \frac{1}{\sqrt{1-(2x)^2}} dx = \frac{1}{2} \text{Sin}^{-1}(2x) + C$$

Anti-differentiation of expressions of the form $f(g(x))g'(x)$ using the substitution $u = g(x)$

Note: $\frac{du}{dx} dx = du$

Example 1 Find $\int 2x\sqrt{1-3x^2} dx$.

$$\begin{aligned} \text{Let } u &= 1-3x^2, \frac{du}{dx} = -6x, \therefore x = -\frac{1}{6} \frac{du}{dx}. \\ \therefore \int 2x\sqrt{1-3x^2} dx &= \int 2\left(-\frac{1}{6}\right)\sqrt{u} \frac{du}{dx} dx \\ &= -\frac{1}{3} \int u^{\frac{1}{2}} du = -\frac{1}{3} \times \frac{2}{3} u^{\frac{3}{2}} + C = -\frac{2}{9} (1-3x^2)^{\frac{3}{2}} + C. \end{aligned}$$

Example 2 Find an anti-derivative of $\frac{(\log_e x)^2}{x}$.

$$\begin{aligned} \text{Let } u &= \log_e x, \frac{du}{dx} = \frac{1}{x}. \\ \int \frac{(\log_e x)^2}{x} dx &= \int u^2 \frac{du}{dx} dx = \int u^2 du \\ &= \frac{1}{3} u^3 + C = \frac{1}{3} (\log_e x)^3 + C. \end{aligned}$$

Example 3 Find $\int \sec^2 x \tan^2 x dx$.

$$\begin{aligned} \text{Let } u &= \tan x, \frac{du}{dx} = \sec^2 x. \\ \int \sec^2 x \tan^2 x dx &= \int u^2 \frac{du}{dx} dx = \int u^2 du \\ &= \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3 x + C. \end{aligned}$$

Example 4 Find $\int \frac{1}{\sqrt{1-x^2} \text{Sin}^{-1} x} dx$.

$$\begin{aligned} \text{Let } u &= \text{Sin}^{-1} x, \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}. \\ \int \frac{1}{\sqrt{1-x^2} \text{Sin}^{-1} x} dx &= \int \frac{1}{u} \frac{du}{dx} dx = \int \frac{1}{u} du \\ &= \log_e |u| + C = \log_e |\text{Sin}^{-1} x| + C \end{aligned}$$

Example 5 Anti-differentiate $\cos^2 x \sin^3 x$.

$$\begin{aligned} \int \cos^2 x \sin^3 x dx &= \int \cos^2 x \sin^2 x \sin x dx \\ &= \int \cos^2 x (1 - \cos^2 x) \sin x dx \\ &= \int (\cos^2 x - \cos^4 x) \sin x dx. \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \cos x, \frac{du}{dx} = -\sin x, \therefore \sin x = -\frac{du}{dx}. \\ \therefore \int (\cos^2 x - \cos^4 x) \sin x dx &= \int (u^2 - u^4) \left(-\frac{du}{dx}\right) dx \\ &= \int (u^4 - u^2) du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C. \end{aligned}$$

Anti-differentiation by linear substitution

This technique is particularly useful for expressions of the form $f(x)g(ax+b)$, where $f(x)$ is a polynomial function and g is a rational power function of $ax+b$.

Using the substitution $u = ax+b$, $f(x)g(ax+b)$ becomes

$$f\left(\frac{u-b}{a}\right)g(u).$$

Example 1 Find $\int x\sqrt{1-x}dx$.

Let $u = 1-x$, then $x = 1-u$, $\frac{du}{dx} = -1$ or $1 = -\frac{du}{dx}$.

$$\begin{aligned}\int x\sqrt{1-x}dx &= \int (1-u)\sqrt{u}\left(-\frac{du}{dx}\right)dx = \int -(1-u)u^{\frac{1}{2}}du \\ &= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C.\end{aligned}$$

Example 2 Find $\int \frac{x^2+1}{\sqrt{x+1}}dx$.

Let $u = x+1$, then $x = u-1$, $\frac{du}{dx} = 1$.

$$\begin{aligned}\int \frac{x^2+1}{\sqrt{x+1}}dx &= \int \frac{(u-1)^2+1}{u^{\frac{1}{2}}}du = \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right)du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 4(x+1)^{\frac{1}{2}} + C.\end{aligned}$$

Example 3 Find an anti-derivative of $(2x+1)(2x-3)^{\frac{2}{3}}$.

Let $u = 2x-3$, then $x = \frac{u+3}{2}$, $\frac{du}{dx} = 2$ or $\frac{1}{2}\frac{du}{dx} = 1$.

$$\begin{aligned}\int (2x+1)(2x-3)^{\frac{2}{3}}dx &= \int (u+4)u^{\frac{2}{3}}\left(\frac{1}{2}\frac{du}{dx}\right)dx \\ &= \frac{1}{2}\int \left(u^{\frac{5}{3}} + 4u^{\frac{2}{3}}\right)du = \frac{1}{2}\left(\frac{3}{8}u^{\frac{8}{3}} + \frac{12}{5}u^{\frac{5}{3}}\right) \\ &= \frac{3}{16}(2x-3)^{\frac{8}{3}} + \frac{6}{5}(2x-3)^{\frac{5}{3}}.\end{aligned}$$

Anti-differentiation using the trigonometric identities

$$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax)) \text{ and } \cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$$

Example 1 Find $\int \cos^2\left(\frac{x}{2}\right)dx$.

$$\begin{aligned}\int \cos^2\left(\frac{x}{2}\right)dx &= \int \frac{1}{2}(1 + \cos x)dx = \frac{1}{2}\int (1 + \cos x)dx \\ &= \frac{1}{2}(x + \sin x + c) = \frac{1}{2}x + \frac{1}{2}\sin x + C.\end{aligned}$$

Example 2 Anti-differentiate $2\sin^2(3x) - 5x$.

$$\begin{aligned}\int (2\sin^2(3x) - 5x)dx &= \int (1 - \cos 2(3x) - 5x)dx \\ &= x - \frac{1}{6}\sin(6x) - \frac{5}{2}x^2 + C.\end{aligned}$$

Example 3 Find an anti-derivative of $\cos^4 x$.

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 = \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \\ &= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \\ &= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \\ &= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) \\ &= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)\end{aligned}$$

$$\begin{aligned}\therefore \int \cos^4 x dx &= \frac{1}{8}\int (3 + 4\cos 2x + \cos 4x)dx \\ &= \frac{1}{8}\left(3x + 2\sin 2x + \frac{1}{4}\sin 4x\right) \\ &= \frac{1}{32}(12x + 8\sin 2x + \sin 4x).\end{aligned}$$

Anti-differentiation using partial fractions of rational functions with quadratic denominators

Partial fractions

A fraction can be changed to the sum or difference of two or more fractions, for example,

$$\frac{5}{6} = \frac{1}{2} + \frac{2}{3}, \quad \frac{3}{4} = \frac{1}{4} + \frac{1}{2}, \quad \frac{11}{10} = 1 + \frac{1}{2} - \frac{4}{5}.$$

Rational algebraic expressions can also be changed to partial fractions if the denominator can be factorised.

Example 1 Change $\frac{x+1}{x^2-4}$ to partial fractions.

$$\frac{x+1}{x^2-4} = \frac{x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}.$$

Compare the numerators: $A(x+2) + B(x-2) = x+1$, it is true for all $x \in R$.

$$\text{Let } x = 2, \quad 4A = 3, \quad A = \frac{3}{4}.$$

$$\text{Let } x = -2, \quad -4B = -1, \quad B = \frac{1}{4}.$$

$$\text{Hence } \frac{x+1}{x^2-4} = \frac{\frac{3}{4}}{x-2} + \frac{\frac{1}{4}}{x+2} \text{ or } \frac{3}{4(x-2)} + \frac{1}{4(x+2)}.$$

Example 2 Convert $\frac{x}{(x-3)^2}$ to partial fractions.

The rational expression has a repeated factor.

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3)+B}{(x-3)^2} \therefore A(x-3)+B=x.$$

Let $x=3$, $\therefore B=3$. Let $x=0$, $\therefore 3A=B=3$, $\therefore A=1$.

$$\text{Hence } \frac{x}{(x-3)^2} = \frac{1}{x-3} + \frac{3}{(x-3)^2}.$$

Example 3 Transform $\frac{2x^2+3}{x^2-3x+2}$ to partial fractions.

The numerator and denominator have the same degree.

$$\begin{aligned} \frac{2x^2+3}{x^2-3x+2} &= \frac{2(x^2-3x+2)+6x-1}{x^2-3x+2} = 2 + \frac{6x-1}{x^2-3x+2} \\ &= 2 + \frac{6x-1}{(x-2)(x-1)} = 2 + \frac{A}{x-2} + \frac{B}{x-1} \\ &= 2 + \frac{A(x-1)+B(x-2)}{(x-2)(x-1)} \therefore A(x-1)+B(x-2)=6x-1. \end{aligned}$$

Let $x=2$, $A=11$. Let $x=1$, $B=-5$.

$$\text{Hence } \frac{2x^2+3}{x^2-3x+2} = 2 + \frac{11}{x-2} - \frac{5}{x-1}.$$

Example 4 Find $\int \frac{x+1}{x^2-4} dx$.

Refer to example 1.

$$\begin{aligned} \int \frac{x+1}{x^2-4} dx &= \int \left(\frac{\frac{3}{4}}{x-2} + \frac{\frac{1}{4}}{x+2} \right) dx \\ &= \frac{3}{4} \log_e|x-2| + \frac{1}{4} \log_e|x+2| + C \end{aligned}$$

Example 5 Find $F(x) = \int f(x) dx$, given $f(x) = \frac{x}{(x-3)^2}$ and $F(4) = 4$.

Refer to example 2, $f(x) = \frac{x}{(x-3)^2} = \frac{1}{x-3} + \frac{3}{(x-3)^2}$.

$$\begin{aligned} F(x) &= \int f(x) dx = \int \left(\frac{1}{x-3} + \frac{3}{(x-3)^2} \right) dx \\ &= \log_e|x-3| - \frac{3}{x-3} + C. \end{aligned}$$

Given $F(4) = 4$, $\therefore F(4) = \log_e|4-3| - \frac{3}{4-3} + C = 4$,

$$\therefore C = 7. \text{ Hence } F(x) = \log_e|x-3| - \frac{3}{x-3} + 7.$$

Example 6 Anti-differentiate $\frac{2x^2+3}{x^2-3x+2}$.

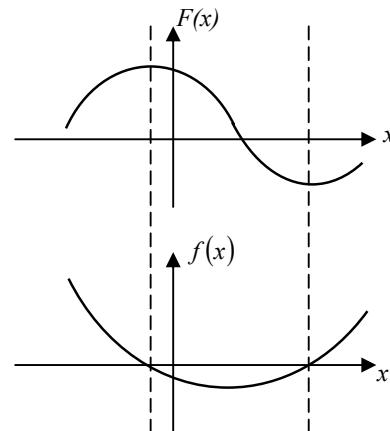
Refer to example 3.

$$\begin{aligned} \int \frac{2x^2+3}{x^2-3x+2} dx &= \int \left(2 + \frac{11}{x-2} - \frac{5}{x-1} \right) dx \\ &= 2x + 11 \log_e|x-2| - 5 \log_e|x-1| + C. \end{aligned}$$

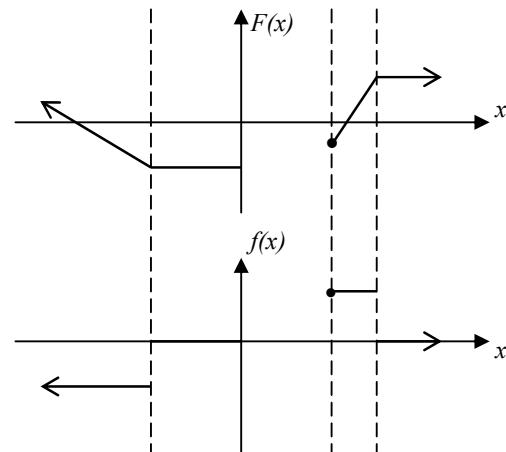
Relationship between the graph of a function and the graphs of its anti-derivatives

The graph of the gradient of an anti-derivative is the graph of the original function, $\because \frac{d}{dx} \int f(x) dx = f(x)$, where $f(x)$ is the original function.

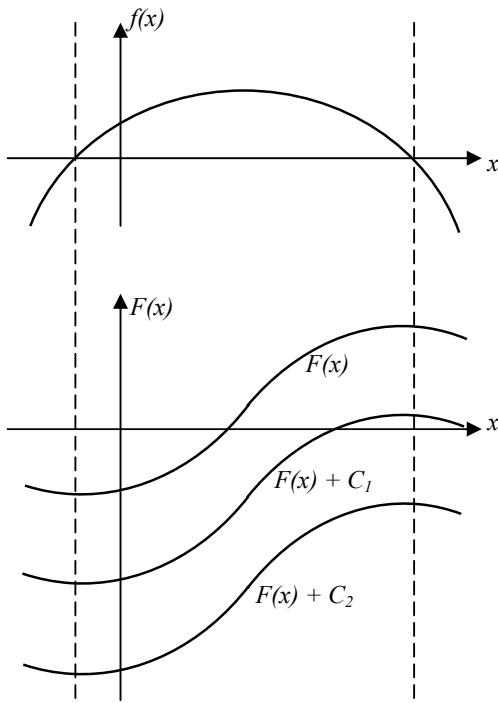
Example 1 Sketch the graph of $f(x)$ given the graph of $F(x) = \int f(x) dx$ shown below.



Example 2 Given the graph of an anti-derivative of a function as shown below, sketch the graph of the function.



Example 3 Sketch the graph of an anti-derivative of the function $f(x)$ shown below.



On the above graph sketch another two anti-derivative graphs of the given function $f(x)$.

Example 3 Evaluate $\int_0^{\frac{\pi}{12}} \cos^5(2x) dx$

$$\begin{aligned} \cos^5(2x) &= (\cos^2(2x))^2 \cos(2x) = (1 - \sin^2(2x))^2 \cos(2x) \\ &= (1 - 2\sin^2(2x) + \sin^4(2x)) \cos(2x). \end{aligned}$$

$$\text{Let } u = \sin(2x), \frac{du}{dx} = 2\cos(2x), \therefore \cos(2x) = \frac{1}{2} \frac{du}{dx}.$$

When $x = 0, u = \sin 0 = 0,$

$$\text{when } x = \frac{\pi}{12}, u = \sin 2\left(\frac{\pi}{12}\right) = \frac{1}{2}.$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{12}} \cos^5(2x) dx &= \int_0^{\frac{\pi}{12}} (1 - 2\sin^2(2x) + \sin^4(2x)) \cos(2x) dx \\ &= \int_0^{\frac{\pi}{12}} (1 - 2u^2 + u^4) \frac{1}{2} \frac{du}{dx} dx \\ &= \int_0^{\frac{1}{2}} \frac{1}{2} (1 - 2u^2 + u^4) du \\ &= \left[\frac{1}{2} \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right) \right]_0^{\frac{1}{2}} \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} \left(\frac{1}{2} \right)^3 + \frac{1}{5} \left(\frac{1}{2} \right)^5 \right) - 0 \\ &= \frac{203}{960}. \end{aligned}$$

Evaluation of definite integrals

In cases involving substitution and change of variable x to u , it is not necessary to convert the anti-derivative back to the original variable x before evaluation. It is simpler to change the values of the lower and upper limits of the definite integral to the values corresponding to the new variable u .

Example 1 Evaluate $\int_1^e \frac{\log_e x}{x} dx$.

$$\text{Let } u = \log_e x, \frac{du}{dx} = \frac{1}{x}.$$

When $x = 1, u = \log_e 1 = 0, \text{ when } x = e, u = \log_e e = 1.$

$$\therefore \int_1^e \frac{\log_e x}{x} dx = \int_0^1 u \frac{du}{dx} dx = \int_0^1 u du = \left[\frac{1}{2} u^2 \right]_0^1 = \frac{1}{2}$$

Example 2 Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx$.

$$\text{Let } u = \tan x, \frac{du}{dx} = \sec^2 x.$$

When $x = 0, u = \tan 0 = 0, \text{ when } x = \frac{\pi}{4}, u = \tan \frac{\pi}{4} = 1.$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx = \int_0^{\frac{\pi}{4}} u \frac{du}{dx} dx = \int_0^1 u du = \left[\frac{1}{2} u^2 \right]_0^1 = \frac{1}{2}.$$