

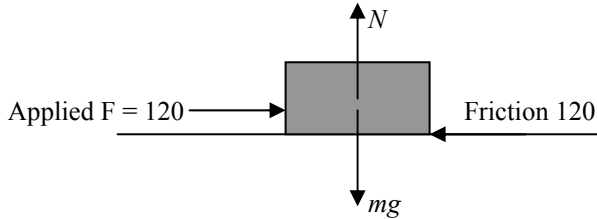
VCAA Specialist Mathematics

Sample exam 1 solutions 2006

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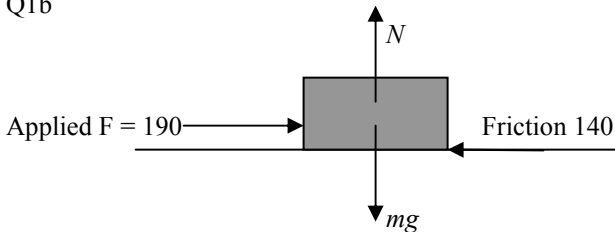
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Q1a Limiting friction = $\mu N = \mu mg = \frac{1}{7}(100)(9.8) = 140$ newtons.



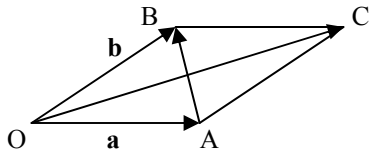
The applied force 120 N is lower than the limiting friction 140 N \therefore friction force is 120 N in the opposite direction to the applied force. Hence the resultant force is zero. Since the crate is at rest, it will remain at rest according to Newton's first law.

Q1b



$$a = \frac{R}{m} = \frac{190 - 140}{100} = 0.5 \text{ ms}^{-2}.$$

Q2



$$\vec{AB} = \mathbf{b} - \mathbf{a}, \vec{OC} = \mathbf{a} + \mathbf{b}.$$

$$\text{Since } \vec{AB} \perp \vec{OC}, \therefore \vec{AB} \cdot \vec{OC} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{a} + \mathbf{b}) = 0.$$

$$\text{Expand to obtain } \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} = 0,$$

$$\therefore \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} = 0 \text{ or } b^2 - a^2 = 0.$$

$$\therefore a = b. \text{ Hence the parallelogram OACB is a rhombus.}$$

Q3a $f(x) = \text{Tan}^{-1}(x) + x\text{Tan}^{-1}(x) = (1+x)\text{Tan}^{-1}(x),$

$$f'(x) = \text{Tan}^{-1}(x) + (1+x) \left(\frac{1}{x^2+1} \right) = \text{Tan}^{-1}(x) + \frac{x+1}{x^2+1}.$$

Q3b At the point of inflection, $f''(x) = 0.$

$$f''(x) = \frac{1}{x^2+1} + \frac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} = \frac{x^2+1+x^2+1-2x^2-2x}{(x^2+1)^2} = 0,$$

$$\therefore 2 - 2x = 0, x = 1, y = (1+1)\text{Tan}^{-1}(1) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}. \therefore \left(1, \frac{\pi}{2}\right).$$

Q4a $2y - xy^2 = 8, \frac{d}{dx}(2y - xy^2) = \frac{d}{dx}(8),$

$$2 \frac{dy}{dx} - \left(y^2 + x(2y) \frac{dy}{dx} \right) = 0. \therefore 2 \frac{dy}{dx} - 2xy \frac{dy}{dx} = y^2,$$

$$\therefore 2(1 - xy) \frac{dy}{dx} = y^2. \text{ Hence } \frac{dy}{dx} = \frac{y^2}{2(1 - xy)}.$$

Q4b When $y = 2, 2(2) - x(2^2) = 8, x = -1.$

$$\therefore \frac{dy}{dx} = \frac{2^2}{2(1 - (-1)2)} = \frac{2}{3}.$$

Q5a $z^3 - 2z^2 + 9z - 18 = (3i)^3 - 2(3i)^2 + 9(3i) - 18$
 $= -27i + 18 + 27i - 18 = 0.$

$\therefore z = 3i$ is a solution of $z^3 - 2z^2 + 9z - 18 = 0.$

Q5b Hence $z - 3i$ is a factor of $z^3 - 2z^2 + 9z - 18$. Since $z^3 - 2z^2 + 9z - 18$ has real coefficients, $\therefore z + 3i$ is also a factor.

$$\therefore z^3 - 2z^2 + 9z - 18 = (z - 3i)(z + 3i)(z - p).$$

$$\therefore -18 = (-3i)(3i)(-p), \therefore p = 2.$$

\therefore the solutions are $3i, -3i, 2.$

Q6a Let $u = \sqrt{3x},$

$$\frac{d}{dx}(\cos^{-1}(u)) = \frac{d}{du}(\cos^{-1}(u)) \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{\sqrt{3}}{2\sqrt{x}}$$

$$= \frac{-\sqrt{3}}{2\sqrt{x}\sqrt{1-3x}} = \frac{-\sqrt{3}}{2\sqrt{x}(1-3x)} = \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}.$$

Q6b $\frac{d}{dx}(\cos^{-1}(\sqrt{3x})) = \frac{-\sqrt{3}}{2\sqrt{x-3x^2}},$

$$\therefore \int_{\frac{1}{6}}^{\frac{1}{4}} \frac{-\sqrt{3}}{2\sqrt{x-3x^2}} dx = \left[\cos^{-1}(\sqrt{3x}) \right]_{\frac{1}{6}}^{\frac{1}{4}} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right),$$

$$\therefore -\frac{\sqrt{3}}{2} \int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x-3x^2}} dx = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12},$$

$$\therefore \int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x-3x^2}} dx = \frac{\sqrt{3}\pi}{18}.$$

Q7

$$y = \int \sin^2(x) \cos^2(x) \sin(x) dx = \int (1 - \cos^2(x)) \cos^2(x) \sin(x) dx$$

$$= \int (\cos^2(x) - \cos^4(x)) \sin(x) dx.$$

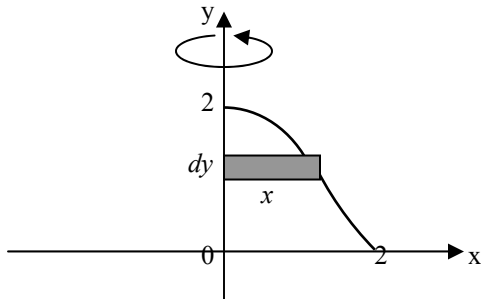
Let $u = \cos(x)$, $\frac{du}{dx} = -\sin(x)$,

$$\therefore y = \int -(\cos^2(x) - \cos^4(x)) \frac{du}{dx} dx, \therefore y = \int (u^4 - u^2) du,$$

$$y = \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + C. \text{ Given } y(0) = 0, \therefore 0 = \frac{1}{5} - \frac{1}{3} + C,$$

$$\therefore C = \frac{2}{15}. \text{ Hence } y = \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + \frac{2}{15}.$$

Q8



$$y = \frac{16}{x^2 + 4} - 2, \therefore x^2 = \frac{16}{y + 2} - 4.$$

$$\text{Volume} = \int_0^2 \pi x^2 dy = \int_0^2 \pi \left(\frac{16}{y + 2} - 4 \right) dy = \pi [16 \log_e |y + 2| - 4y]_0^2$$

.For $-2 \leq x \leq 2$, $y \geq 0$, $\therefore y + 2 > 0$,

$$\therefore \text{volume} = \pi [16 \log_e (y + 2) - 4y]_0^2$$

$$= \pi (16 \log_e 4 - 8) - \pi 16 \log_e 2 = 8\pi (\log_e 4 - 1).$$

Q9a The discriminant of $x^2 + 4x + 5$ is $\Delta = 4^2 - 4(1)(5) = -4$,

$\therefore x^2 + 4x + 5 > 0$ for all x . Hence $y = \frac{3}{x^2 + 4x + 5}$ is defined

for all x .

Q9b $\int \frac{3}{x^2 + 4x + 5} dx = \int \frac{3}{1 + (x + 2)^2} dx = 3 \tan^{-1}(x + 2) + c.$

$\therefore A = 3, B = 2.$

Q9c The turning point is $(-2, 1)$. $\therefore a = -2.$

$$\text{Area} = \int_{-2}^{\sqrt{3}-2} \frac{3}{x^2 + 4x + 5} dx = [3 \tan^{-1}(x + 2)]_{-2}^{\sqrt{3}-2}$$

$$= 3 \tan^{-1}(\sqrt{3}) - 3 \tan^{-1}(0) = \pi.$$

Q9d Consider $f(x) = 5 \tan^{-1}(x + 7)$, it has a range of

$$\left(-\frac{5\pi}{2}, \frac{5\pi}{2} \right).$$

$$\therefore -\frac{5\pi}{2} < 5 \tan^{-1}(x + 7) \text{ for all } x, \therefore 0 < 5 \tan^{-1}(x + 7) + \frac{5\pi}{2}.$$

Hence $c > \frac{5\pi}{2}.$

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