

Q1a $8x^3 + 27 = (2x)^3 + 3^3 = (2x+3)(4x^2 - 6x + 9)$

Q1b $f(x) = \ln(x-3)$, $x-3 > 0$, $x > 3$. Domain is $(3, \infty)$.

Q1c $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cos x \rightarrow 2 \times 1 = 2$.

Q1d $\frac{x+3}{2x} > 1$, $\frac{1}{2} + \frac{3}{2x} > 1$, $\frac{3}{2x} > \frac{1}{2}$, $\frac{1}{x} > \frac{1}{3}$, $\therefore 0 < x < 3$.

Q1e $\frac{d}{dx}(x \cos^2 x) = (1) \cos^2 x + x(-2 \cos x \sin x) = \cos^2 x - x \sin 2x$.

Q1f $u = x^3 + 1$, $\frac{1}{3} \frac{du}{dx} = x^2$.

$\int_0^2 x^2 e^{x^3+1} dx = \int_0^2 \frac{1}{3} e^u \frac{du}{dx} dx = \int_1^9 \frac{1}{3} e^u du = \left[\frac{1}{3} e^u \right]_1^9 = \frac{e}{3}(e^8 - 1)$.

Q2a $p(x) = x^3 - ax + b$.

Divided by $x-1$, $R = p(1) = 1 - a + b = 2$, $\therefore -a + b = 1$ (1)

Divided by $x+2$, $R = p(-2) = -8 + 2a + b = 5$,

$\therefore 2a + b = 13$ (2)

(2) - (1): $3a = 12$, $\therefore a = 4$ and $b = 5$.

Q2bi Let $3 \sin x + 4 \cos x = A \sin(x + \alpha)$.

$\therefore 3 \sin x + 4 \cos x = A \sin x \cos \alpha + A \cos x \sin \alpha$.

Hence, $A \cos \alpha = 3$ (1)

and $A \sin \alpha = 4$ (2)

$\therefore (A \cos \alpha)^2 + (A \sin \alpha)^2 = 25$, $A^2(\cos^2 \alpha + \sin^2 \alpha) = 25$, $A = 5$.

$\therefore \sin \alpha = \frac{4}{5}$, $\alpha = \sin^{-1} 0.8$.

Hence $3 \sin x + 4 \cos x = 5 \sin(x + \sin^{-1} 0.8)$.

Q2bii $3 \sin x + 4 \cos x = 5$, $5 \sin(x + \sin^{-1} 0.8) = 5$,

$\sin(x + \sin^{-1} 0.8) = 1$.

$\therefore x + \sin^{-1} 0.8 = \frac{\pi}{2}$, $x \approx 0.64$.

Q2ci At $P(2t, t^2)$, $\frac{dy}{dx} = \frac{x}{2} = t$.

Equation of tangent at $(2t, t^2)$: $y - t^2 = t(x - 2t)$,

$\therefore y = tx - t^2$ (1)

Q2cii At $Q(4t, 4t^2)$, i.e. $Q(2(2t), (2t)^2)$, equation of tangent is $y = (2t)x - (2t)^2$, $\therefore y = 2tx - 4t^2$ (2)

Solve (1) and (2) simultaneously to find intersection R .

$\therefore 2tx - 4t^2 = tx - t^2$, $tx - 3t^2 = 0$, $t(x - 3t) = 0$.

For the situation as shown in the diagram, $t \neq 0$, $\therefore x = 3t$ and $y = 2t^2$. Hence $R(3t, 2t^2)$, where $t \neq 0$.

Note: No unique intersection when $t = 0$.

Q2ciii $R(3t, 2t^2)$, $x = 3t$, $\therefore t = \frac{x}{3}$.

$y = 2t^2$, $\therefore y = 2\left(\frac{x}{3}\right)^2$, i.e. $y = \frac{2}{9}x^2$, $x \neq 0$.

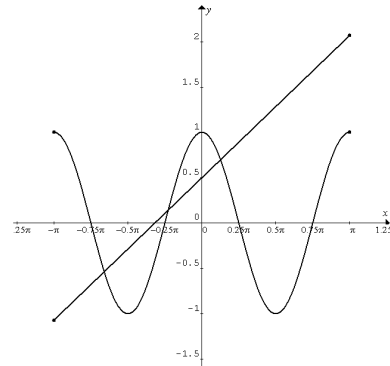
Q3ai $f(x) = \frac{3+e^{2x}}{4} = \frac{1}{4}e^{2x} + \frac{3}{4}$. The range is $\left(\frac{3}{4}, \infty\right)$.

Q3aii Equation of inverse: $x = \frac{3+e^{2y}}{4}$, $4x - 3 = e^{2y}$,

$y = \frac{1}{2} \ln(4x - 3)$ or $y = \ln \sqrt{4x - 3}$.

$\therefore f^{-1}(x) = \ln \sqrt{4x - 3}$.

Q3bi



Q3bii $2 \cos 2x = x + 1$, $\cos 2x = \frac{x+1}{2}$. The solutions to this equation are the x -coordinates of the intersections of the graphs of $y = \cos 2x$ and $y = \frac{x+1}{2}$. There are 3 intersections, $\therefore 3$ solutions.

Q3biii $f(x) = 2 \cos 2x - (x+1)$, $f'(x) = -4 \sin 2x - 1$.

Newton's method: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

$x_1 = 0.4$

$x_2 = 0.4 - \frac{2 \cos 0.8 - 1.4}{-4 \sin 0.8 - 1} \approx 0.398$.

$$\begin{aligned} \text{Q3ci } R.H.S. &= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{1 - (1 - 2\sin^2 \theta)}{1 + (2\cos^2 \theta - 1)} \\ &= \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta = L.H.S. \end{aligned}$$

$$\text{Q3cii } \tan \theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}},$$

$$\begin{aligned} \tan \frac{\pi}{8} &= \sqrt{\frac{1 - \cos \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}}} = \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} = \sqrt{\frac{(\sqrt{2} - 1)(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)}} \\ &= \sqrt{(\sqrt{2} - 1)^2} = \sqrt{2} - 1. \end{aligned}$$

$$\text{Q4ai Binomial: } n = 5, p = \frac{1}{4}, q = \frac{3}{4}.$$

$$\Pr(X = 3) = \binom{5}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 = \frac{45}{512}.$$

$$\begin{aligned} \text{Q4aii } \Pr(X \geq 3) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) \\ &= \binom{5}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 + \binom{5}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^1 + \binom{5}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0 = \frac{53}{512}. \end{aligned}$$

Q4aiii Let X' be the number of incorrect answers.

$$\Pr(X' \geq 1) = 1 - \Pr(X' = 0) = 1 - \binom{5}{0} \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 = \frac{1023}{1024}.$$

$$\text{Q4bi } f(x) = \frac{x^4 + 3x^2}{x^4 + 3}.$$

$$f(-x) = \frac{(-x)^4 + 3(-x)^2}{(-x)^4 + 3} = \frac{x^4 + 3x^2}{x^4 + 3} = f(x).$$

$\therefore f(x)$ is an even function.

$$\text{Q4bii } f(x) = \frac{x^4 + 3x^2}{x^4 + 3} = 1 + \frac{3x^2 - 3}{x^4 + 3}.$$

Horizontal asymptote is $y = 1$.

$$\text{Q4biii } f(x) = 1 + \frac{3x^2 - 3}{x^4 + 3},$$

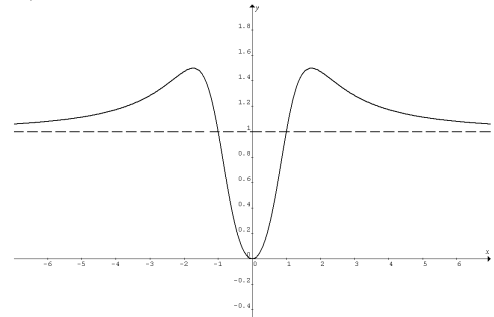
$$f'(x) = \frac{(x^4 + 3)(6x) - (3x^2 - 3)(4x^3)}{(x^4 + 3)^2} = \frac{18x + 12x^3 - 6x^5}{(x^4 + 3)^2} = 0.$$

$$\therefore 18x + 12x^3 - 6x^5 = 0, 6x(3 + 2x^2 - x^4) = 0,$$

$$6x(3 - x^2)(1 + x^2) = 0, 6x(\sqrt{3} - x)(\sqrt{3} + x)(1 + x^2) = 0.$$

Since $1 + x^2 \neq 0$, $\therefore x = 0, \pm\sqrt{3}$ are the x -coordinates of the stationary points.

Q4biv When $x = 0$, $y = f(0) = 0$. When $x = \pm\sqrt{3}$, $y = f(\pm\sqrt{3}) = 1.5$.



$$\text{Q5ai } \frac{d^2x}{dt^2} = -n^2x, \text{ or } \frac{1}{2} \frac{d(v^2)}{dx} = -n^2x, \frac{d(v^2)}{dx} = -2n^2x,$$

$$v^2 = \int -2n^2x dx = -n^2x^2 + c.$$

At $x = a$, $v = 0$. $\therefore c = n^2a^2$ and $v^2 = n^2(a^2 - x^2)$.

Q5aii Speed is maximum at $x = 0$, $v^2 = n^2a^2$, $\therefore |v|_{\max} = na$.

Q5aiii |Acceleration| is maximum at $x = a$, $\therefore \left| \frac{d^2x}{dt^2} \right|_{\max} = n^2a$.

Q5aiv Consider $v = n\sqrt{a^2 - x^2}$. Note: $v = -n\sqrt{a^2 - x^2}$ will yield the same result.

$$\frac{dx}{dt} = n\sqrt{a^2 - x^2}, \frac{dt}{dx} = \frac{1}{n\sqrt{a^2 - x^2}}, t = \int \frac{1}{n\sqrt{a^2 - x^2}} dx,$$

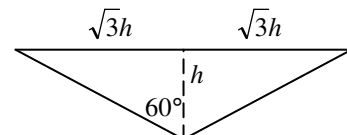
$$nt = \sin^{-1} \frac{x}{a} + c. \text{ When } t = 0, x = 0, \therefore c = 0 \text{ and } nt = \sin^{-1} \frac{x}{a}.$$

Hence $x = a \sin nt$.

Now $v = \frac{dx}{dt} = a n \cos nt = \frac{na}{2}$, $\cos nt = \frac{1}{2}$, $nt = \frac{\pi}{3}$, $\therefore t = \frac{\pi}{3n}$ is

the first time when the particle's speed is half its maximum speed.

Q5bi



$$\text{Area of cross-section} = \frac{1}{2} \times 2\sqrt{3}h \times h = \sqrt{3}h^2 \text{ m}^2.$$

$$\text{Volume of water } V = 10\sqrt{3}h^2 \text{ m}^3.$$

Q5bii Top water surface area $A = 10 \times 2\sqrt{3}h = 20\sqrt{3}h \text{ m}^2$.

Q5biii $\frac{dV}{dt} = -kA$, $\frac{dV}{dh} \times \frac{dh}{dt} = -kA$, $20\sqrt{3}h \frac{dh}{dt} = -k \times 20\sqrt{3}h$,
 $\therefore \frac{dh}{dt} = -k$, where $k > 0$. Note that the depth of water decreases at a constant rate.

Q5biv The constant rate is 1 m per 100 days. \therefore it takes another 100 days to fall from 2 m to 1 m.

Q6ai At time $t = T$, $x_1 = x_2$. $\therefore UT \cos \theta = R - VT \cos \theta$,
 $UT \cos \theta + VT \cos \theta = R$, $T(U + V) \cos \theta = R$,
 $\therefore T = \frac{R}{(U + V) \cos \theta}$.

Q6aii At time $t = T$,
 $y_1 = UT \sin \theta - \frac{1}{2}gT^2$ and $y_2 = h - VT \sin \theta - \frac{1}{2}gT^2$.
 $h - VT \sin \theta = R \tan \theta - VT \sin \theta = \frac{R \sin \theta}{\cos \theta} - \frac{VR \sin \theta}{(U + V) \cos \theta}$
 $= \frac{(U + V)R \sin \theta - VR \sin \theta}{(U + V) \cos \theta} = \frac{UR \sin \theta}{(U + V) \cos \theta} = UT \sin \theta$.
 $\therefore y_1 = y_2$. Hence the projectiles collide.

Q6aiii At time $t = T$, $x = \lambda R$ and $x = UT \cos \theta$, where $0 < \lambda < 1$.

$\therefore \lambda R = UT \cos \theta$, $\lambda R = \frac{UR}{U + V}$, $\lambda = \frac{U}{U + V}$, $\lambda U + \lambda V = U$.

Hence $V = \frac{U - \lambda U}{\lambda} = \left(\frac{1 - \lambda}{\lambda}\right)U = \left(\frac{1}{\lambda} - 1\right)U$.

Q6bi
 $(1 + x)^r + (1 + x)^{r+1} + \dots + (1 + x)^n = (1 + x)^r [1 + (1 + x) + \dots + (1 + x)^{n-r}]$
 $= (1 + x)^r \left[\frac{1 - (1 + x)^{n-r+1}}{(1 + x) - 1} \right] = (1 + x)^r \left[\frac{(1 + x)^{n-r+1} - 1}{x} \right]$
 $= \frac{(1 + x)^{n+1} - (1 + x)^r}{x}$.
 $\therefore (1 + x)^r + (1 + x)^{r+1} + \dots + (1 + x)^n = \frac{(1 + x)^{n+1}}{x} - \frac{(1 + x)^r}{x}$

Consider only the coefficients of the x^r terms in the expression

above: $\binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n}{r} = \binom{n+1}{r+1}$.

Note that for $(1 + x)^{n+1}$, the coefficient of x^{r+1} is $\binom{n+1}{r+1}$.

\therefore for $\frac{(1 + x)^{n+1}}{x}$, the coefficient of x^r is $\binom{n+1}{r+1}$. Also $\frac{(1 + x)^r}{x}$

does not have an x^r term.

Q6bii(1) There are n points on the line $y = x$. Select any 2 points to form an interval, the number of intervals is $\binom{n}{2}$.

Q6bii(2) For 1 point, number of intervals is 0.

For 2 points, number of intervals is $\binom{2}{2}$.

For 3 points, number of intervals is $\binom{3}{2}$.

For $n-1$ points, number of intervals is $\binom{n-1}{2}$.

For n points, number of intervals is $\binom{n}{2}$.

$\therefore S_n = \binom{2}{2} + \binom{3}{2} + \dots + \binom{n-1}{2} + \binom{n}{2} = \binom{n-1}{2} + \dots + \binom{3}{2} + \binom{2}{2}$.

Q6biii $\therefore S_n = 2 \left[\binom{2}{2} + \binom{3}{2} + \dots + \binom{n-1}{2} \right] + \binom{n}{2}$.

Using the result in Q6bi, $S_n = 2 \left[\binom{n-1+1}{2+1} + \binom{n}{2} \right] = 2 \left[\binom{n}{3} + \binom{n}{2} \right]$
 $= 2 \times \frac{n(n-1)(n-2)}{3 \times 2 \times 1} + \frac{n(n-1)}{2 \times 1} = \frac{2n(n-1)(n-2) + 3n(n-1)}{6}$
 $= \frac{n(n-1)[2(n-2) + 3]}{6} = \frac{n(n-1)(2n-1)}{6}$.

Q7ai $\lim_{h \rightarrow 0} \frac{(x+h) - x}{h} = \lim_{h \rightarrow 0} 1 = 1$.

Q7aii $\frac{d}{dx}(x) = 1$, \therefore the statement is true for $n = 1$.

Assume it is true for $n = k$, i.e. $\frac{d}{dx}(x^k) = kx^{k-1}$.

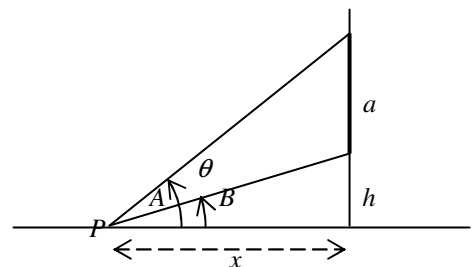
Now consider $n = k + 1$.

$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k) = (x)(kx^{k-1}) + (1)(x^k) = (k+1)x^k$.

\therefore the statement is true for $n = k + 1$.

\therefore the statement is true for all positive integers n .

Q7bi



From the diagram, $\theta = A - B$, $\tan A = \frac{a+h}{x}$, $\tan B = \frac{h}{x}$.

$$\tan \theta = \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{a+h}{x} - \frac{h}{x}}{1 + \frac{a+h}{x} \times \frac{h}{x}} = \frac{\frac{a}{x}}{1 + \frac{h(a+h)}{x^2}}$$

$$= \frac{ax}{x^2 + h(a+h)}. \text{ Hence } \theta = \tan^{-1} \left[\frac{ax}{x^2 + h(a+h)} \right].$$

Q7bii $\tan \theta = \frac{ax}{x^2 + h(a+h)}$, $\frac{d}{dx}(\tan \theta) = \frac{d}{dx} \left[\frac{ax}{x^2 + h(a+h)} \right]$,

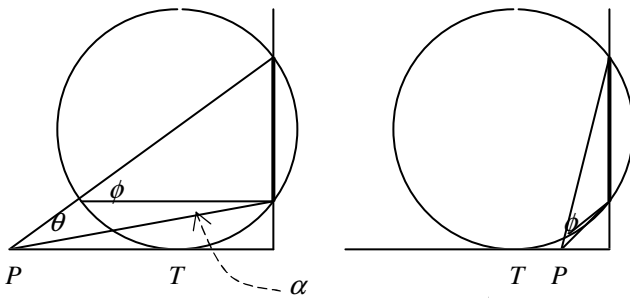
$$\sec^2 \theta \times \frac{d\theta}{dx} = \frac{(x^2 + h(a+h))(a) - (ax)(2x)}{(x^2 + h(a+h))^2},$$

$$\frac{d\theta}{dx} = \frac{a(x^2 + h(a+h)) - 2ax^2}{(x^2 + h(a+h))^2 \sec^2 \theta}.$$

$$\frac{d\theta}{dx} = 0 \text{ when } a(x^2 + h(a+h)) - 2ax^2 = 0.$$

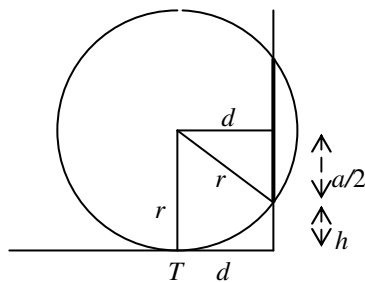
$\therefore x^2 = h(a+h)$ and $x = \sqrt{h(a+h)}$ for maximum θ .

Q7ci



In both cases, $\theta + \alpha = \phi$, $\therefore \theta < \phi$ when P and T are different points. When P and T are the same point, $\theta = \phi$ is a maximum.

Q7cii



Radius of the circle $r = \frac{a}{2} + h$.

Distance of T from the building :

$$d = \sqrt{\left(\frac{a}{2} + h\right)^2 - \left(\frac{a}{2}\right)^2} = \sqrt{h(a+h)}.$$

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