2010 Specialist Mathematics Trial Exam 1 Solutions

Q1 $4z^3 - i2\sqrt{3}z^2 - 3z = 0$, $z\left(4z^2 - i2\sqrt{3}z - 3\right) = 0$

$\therefore z = 0$ or $z = \dfrac{i2\sqrt{3} \pm \sqrt{\left(-i2\sqrt{3}\right)^2 - 4(4)(-3)}}{2(4)} = \dfrac{i\sqrt{3} \pm 3}{4}$, i.e.

$z = 0$, $\dfrac{3}{4} + \dfrac{\sqrt{3}}{4}i$ or $-\dfrac{3}{4} + \dfrac{\sqrt{3}}{4}i$

Q2 $\left\{z : |z+4| + \left|z - 3\sqrt{3}\right| = 11\right\} \cap \{z : \operatorname{Re}(z) = 0\}$

$\sqrt{(x+4)^2 + y^2} + \sqrt{\left(x - 3\sqrt{3}\right)^2 + y^2} = 11$ and $x = 0$

$\therefore \sqrt{16 + y^2} + \sqrt{27 + y^2} = 11$

$\left(\sqrt{27 + y^2}\right)^2 = \left(11 - \sqrt{16 + y^2}\right)^2$

$27 + y^2 = 121 - 2 \times 11\sqrt{16 + y^2} + 16 + y^2$

$16 + y^2 = 25$, $\therefore y = \pm 3$

$\therefore \{-3i, 3i\}$

Q3a $T = \dfrac{2\pi}{\frac{1}{10}} = 20\pi$ seconds

Q3b Radius $= 3$ m, circumference $= 6\pi$ m,

speed $= \dfrac{6\pi}{20\pi} = 0.3$ ms^{-1}

Q4 $a = {}^-3$, $u = {}^+8.5$, $s = {}^+3 - {}^-2 = {}^+5$, t? Use $s = ut + \dfrac{1}{2}at^2$,

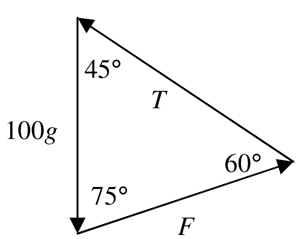
$5 = 8.5t - 1.5t^2$, $t = \dfrac{2}{3}$ or $t = 5$.

When $t = \dfrac{2}{3}$, $v = u + at = 8.5 - 3 \times \dfrac{2}{3} = {}^+6.5$, i.e. same direction as the initial velocity.

When $t = 5$, $v = 8.5 - 3 \times 5 = {}^-6.5$, i.e. in the direction opposite to the initial velocity.

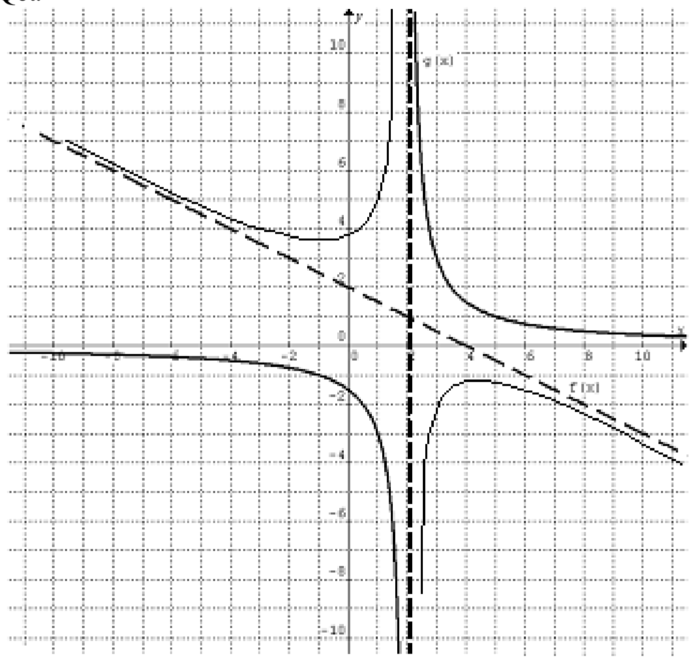
$\therefore t = 5$ s

Q5 The force vectors form a triangle.



$\dfrac{F}{\sin 45°} = \dfrac{100g}{\sin 60°}$, $F = \dfrac{100\sqrt{6}g}{3}$ newtons

Q6a



Q6b $x = 2$, $y = -\dfrac{1}{2}x + 2$

Q7a $RHS = \sec x(\sec x + \tan x) = \dfrac{1}{\cos x}\left(\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}\right)$

$= \dfrac{1 + \sin x}{\cos^2 x} = \dfrac{1 + \sin x}{1 - \sin^2 x} = \dfrac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$

$= \dfrac{1}{1 - \sin x} = LHS$

Q7b $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin x}dx = \int_0^{\frac{\pi}{3}} \sec x(\sec x + \tan x)dx$

$= \displaystyle\int_0^{\frac{\pi}{3}} \left(\sec^2 x + \sec x \tan x\right)dx$

$= \displaystyle\int_0^{\frac{\pi}{3}} \left(\sec^2 x + \frac{\sin x}{\cos^2 x}\right)dx$

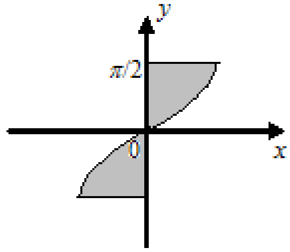
$= \displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x dx + \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x}dx$

$= [\tan x]_0^{\frac{\pi}{3}} + \left[\dfrac{1}{u}\right]_1^{\frac{1}{2}}$ (where $u = \cos x$)

$= \tan\dfrac{\pi}{3} + 2 - 1$

$= \sqrt{3} + 1$

Q8a



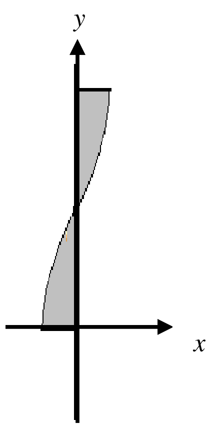
$$Area = 2\times\int_0^{\frac{\pi}{2}} x\,dy = 2\times\int_0^{\frac{\pi}{2}} \sin y\,dy$$

$$= 2\times\left[-\cos y\right]_0^{\frac{\pi}{2}} = 2$$

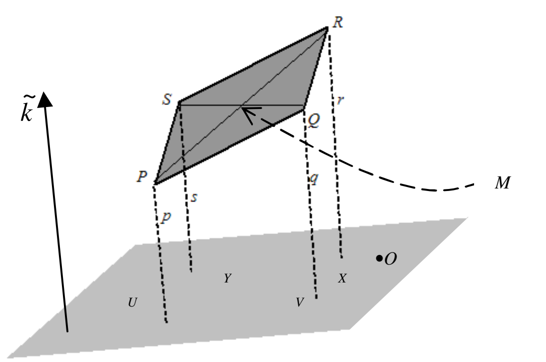
Q8b The graph of $y = 2\sin^{-1}(2x)+\pi$ is the graph of $y = \sin^{-1} x$ after the following transformations:

(1) Vertical dilation by a factor of 2
(2) Horizontal dilation by a factor of ½
(3) Upward translation by π units

After transformation (1) the area of the shaded region is doubled, and after transformation (2) the area is halved. Thus the area of the shaded region remains the same. Transformation (3) does not affect the area of the shaded region.
Shaded area $= 2$



Q9a



Let *M* be the intersection of the two diagonals of the rectangle, and *O* a point on the flat surface.

$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OR}\right) = \frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OS}\right)$ because *M* is the midpoint of the two diagonals. Hence $\overrightarrow{OP}+\overrightarrow{OR} = \overrightarrow{OQ}+\overrightarrow{OS}$.

Q9b Let *U*, *V*, *X* and *Y* be the end points of *p*, *q* *r* and *s* on the flat surface as shown in the previous diagram.
Unit vector $\tilde{k}$ is perpendicular to the flat surface as shown.
Since $\overrightarrow{OP}+\overrightarrow{OR} = \overrightarrow{OQ}+\overrightarrow{OS}$,
$\therefore \left(\overrightarrow{OU}+\overrightarrow{UP}\right)+\left(\overrightarrow{OX}+\overrightarrow{XR}\right) = \left(\overrightarrow{OV}+\overrightarrow{VQ}\right)+\left(\overrightarrow{OY}+\overrightarrow{YS}\right)$
$\therefore \left(\overrightarrow{OU}+\overrightarrow{OX}\right)+\left(\overrightarrow{UP}+\overrightarrow{XR}\right) = \left(\overrightarrow{OV}+\overrightarrow{OY}\right)+\left(\overrightarrow{VQ}+\overrightarrow{YS}\right)$
$\therefore \left(\overrightarrow{OU}+\overrightarrow{OX}\right)+(p+r)\tilde{k} = \left(\overrightarrow{OV}+\overrightarrow{OY}\right)+(q+s)\tilde{k}$
$\tilde{k}\,\bullet$ both sides:
$\left(\overrightarrow{OU}+\overrightarrow{OX}\right)\bullet\tilde{k} + (p+r)\tilde{k}\bullet\tilde{k} = \left(\overrightarrow{OV}+\overrightarrow{OY}\right)\bullet\tilde{k} + (q+s)\tilde{k}\bullet\tilde{k}$
Since $\left(\overrightarrow{OU}+\overrightarrow{OX}\right)$ and $\left(\overrightarrow{OV}+\overrightarrow{OY}\right)$ are vectors on the flat surface,
$\therefore \left(\overrightarrow{OU}+\overrightarrow{OX}\right)\bullet\tilde{k} = \left(\overrightarrow{OV}+\overrightarrow{OY}\right)\bullet\tilde{k} = 0$. Hence $p+r = q+s$.

Q10a $\left(x^2+y^2\right)^2 = x^2-y^2$

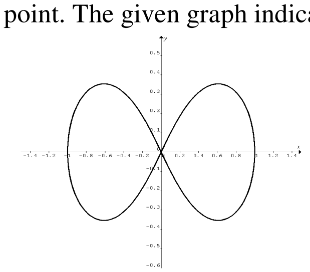
Implicit differentiation, $2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) = 2x-2y\frac{dy}{dx}$.

Expand, $4x\left(x^2+y^2\right)+4y\left(x^2+y^2\right)\frac{dy}{dx} = 2x-2y\frac{dy}{dx}$.

Regroup, $\left(2y+4y\left(x^2+y^2\right)\right)\frac{dy}{dx} = 2x-4x\left(x^2+y^2\right)$.

$\therefore \frac{dy}{dx} = \frac{x\left(1-2\left(x^2+y^2\right)\right)}{y\left(1+2\left(x^2+y^2\right)\right)}$.

Q10b The tangents are horizontal at the points where $\frac{dy}{dx} = 0$.
$\therefore x\left(1-2\left(x^2+y^2\right)\right) = 0$ and $y\left(1+2\left(x^2+y^2\right)\right) \neq 0$.
$\therefore (0,0)$ is not the point. The given graph indicates the same.



$\therefore 1-2\left(x^2+y^2\right) = 0$, i.e. $x^2+y^2 = \frac{1}{2}$ …… (1)

From $\left(x^2+y^2\right)^2 = x^2-y^2$, $x^2-y^2 = \frac{1}{4}$ …… (2)

Eq (1) + eq (2): $2x^2 = \frac{3}{4}$, $\therefore x = \frac{\sqrt{6}}{4}$ and $y = \pm\frac{\sqrt{2}}{4}$,

or $\therefore x = -\frac{\sqrt{6}}{4}$ and $y = \pm\frac{\sqrt{2}}{4}$.

The points are $\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$, $\left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$, $\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$ and $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

