2014 VCAA Specialist Math Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
B	E	D	C	A	D	E	C	B	A	D

12	13	14	15	16	17	18	19	20	21	22
C	E	C	B	A	D	E	C	B	A	E

Q1 Asymptotes are $y=\pm\frac{2}{3}(x-3)$

x-intercept: $(3,0)$; y-intercepts: $(0,-2)$ and $(0,2)$ B

Q2 $x^2-6x+2y^2+8y+16=0$,

$x^2-6x+9+2(y^2+4y+4)+16=9+8$

$(x-3)^2+2(y+2)^2=1$, $\frac{(x-3)^2}{1^2}+\frac{(y+2)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}=1$ E

Q3 $f(x)=\frac{x^2-4x+3}{x^2-x-6}=1-\frac{3(x-3)}{(x+2)(x-3)}$

or $1-\frac{3}{x+2}$ where $x\neq 3$ D

Q4 For $\arcsin(2x-1)$, $-1\le 2x-1\le 1$, .: $0\le x\le 1$ C

Q5 $z^2=\left(2\sqrt{2}\right)^2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-8i$ A

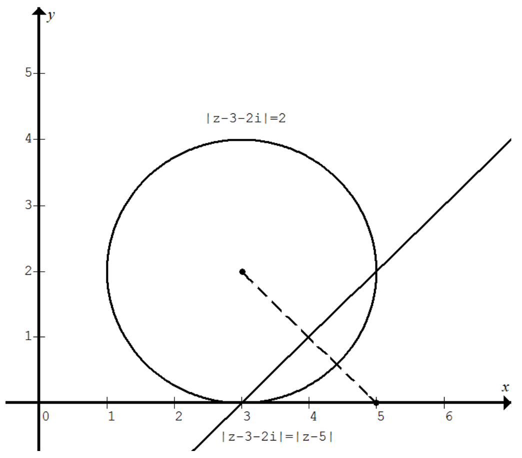
Q6 $i^{2n+3}=i^{2n}i^2i=\left(i^n\right)^2i^2i=-ip^2$ D

Q7 $z^3-5z^2+11z-7=(z-\alpha)(z-\beta)(z-\gamma)=0$

Coefficient of z^2: $-(\alpha+\beta+\gamma)=-5$, .: $\alpha+\beta+\gamma=5$ E

Q8 $\frac{-3\sqrt{2}-i\sqrt{6}}{2+2i}=\frac{A\,\text{cis}\,\frac{7\pi}{6}}{B\,\text{cis}\,\frac{\pi}{4}}=C\,\text{cis}\,\frac{11\pi}{12}$ C

Q9 B



Q10 After t minutes, Q kg of salt is in $1500-2t$ litres of solution, .: concentration is $\frac{Q}{1500-2t}$ kg per litre

Rate of inflow $=2\times 8=16$ kg per minute (2 kg of salt per litre?)

Rate of outflow $=\frac{Q}{1500-2t}\times 10=\frac{5Q}{750-t}$ kg per minute

.: $\frac{dQ}{dt}=16-\frac{5Q}{750-t}$ A

Q11 $y_{n+1}\approx y_n+h\times\frac{dy}{dx}$

$x_0=1$ $\quad y_0=2$ $\quad \frac{dy}{dx}=x^3-xy=-1$

$x_1=1.1$ $\quad y_1\approx 2+0.1(-1)=1.9$ D

Q12 C

Q13 $u=\sqrt{x+1}$, $\frac{du}{dx}=\frac{1}{2\sqrt{x+1}}$, $u^2+1=x+2$

When $x=0$, $u=1$; when $x=2$, $u=\sqrt{3}$

$\int_0^2\frac{dx}{(x+2)\sqrt{x+1}}=2\int_1^{\sqrt{3}}\frac{1}{u^2+1}\,du$ E

Q14 At $y=x$, $\frac{dy}{dx}$ is undefined; when $y<x$, $\frac{dy}{dx}<0$;

when $y>x$, $\frac{dy}{dx}>0$ C

Q15 $|\tilde{a}|=\sqrt{20}$, $|\tilde{b}|=\sqrt{20}$, $\cos\theta=\frac{\tilde{a}.\tilde{b}}{|\tilde{a}||\tilde{b}|}=-\frac{4}{5}$

$\cos 2\theta=2\cos^2\theta-1=\frac{7}{25}$ B

Q16 $\tilde{a}=4\tilde{i}+m\tilde{j}-3\tilde{k}$, $\tilde{b}=-2\tilde{i}+n\tilde{j}-\tilde{k}$, where m, $n\in R^+$

$|\tilde{a}|^2=100$, .: $m=5\sqrt{3}$; $\tilde{a}.\tilde{b}=0$, .: $mn=5$, .: $n=\frac{\sqrt{3}}{3}$ A

Q17 $\tilde{v}(t)=\int\left(-4\sin 2t\,\tilde{i}+20\cos 2t\,\tilde{j}-20e^{-2t}\,\tilde{k}\right)dt$

$=2\cos 2t\,\tilde{i}+10\sin 2t\,\tilde{j}+10e^{-2t}\,\tilde{k}+\tilde{c}$

Given $\tilde{v}(0)=0$, .: $\tilde{c}=-2\tilde{i}-10\tilde{k}$ and

$\tilde{v}(t)=(2\cos 2t-2)\tilde{i}+10\sin 2t\,\tilde{j}+\left(10e^{-2t}-10\right)\tilde{k}$ D

Q18 North-south: $1+2\cos 60°+4\cos 120°=0$

East-west: $2\sin 60°+4\sin 120°-5=3\sqrt{3}-5>0$

.: the net force acts in a easterly direction.

The initial state of motion is not specified!

Assume that the body is initially at rest (or moving to the east), it will move to the east. E itute.com

Q19 $\tilde{v}(t) = 3\sin 2t\,\tilde{i} + 4\cos 2t\,\tilde{j}$, $\tilde{a}(t) = \frac{d\tilde{v}}{dt} = 6\cos 2t\,\tilde{i} - 8\sin 2t\,\tilde{j}$

Net force $= m\tilde{a} = 30\cos 2t\,\tilde{i} - 40\sin 2t\,\tilde{j}$

|Net force| $= \sqrt{900\cos_2 2t + 1600\sin^2 2t} = \sqrt{900 + 700\sin^2 2t}$

.: the max. magnitude of the net force $= \sqrt{900+700} = 40$ C

Q20 Net force $= 5\times 9.8 - 3\times 9.8 = 19.6$ N

Acceleration $= \frac{\text{net force}}{\text{total mass}} = \frac{19.6}{8} = 2.45$ m s^{-2}

After 2 seconds, $v = u + at = 0 + 2.45\times 2 = 4.9$ m s^{-1} B

Q21 $a = -4x$, $\frac{1}{2}\frac{d\,v^2}{dx} = -4x$, $\frac{d\,v^2}{dx} = -8x$, $v^2 = -4x^2 + c$

Given $v = 0$ at $x = 5$, .: $c = 100$ and $v^2 = 100 - 4x^2$

At $x = 3$, $v^2 = 64$, .: $|v| = 8$ A

Q22 In $0 \le t \le 4$, distance $= \frac{1}{2}(2+4)(9) = 27$

In $4 \le t \le 8$, distance $= \int_4^8 \left(-\frac{9}{16}(t-4)^2 + 9\right)dt = 24$

In $8 \le t \le 9$, distance $= -\int_8^9 \left(-\frac{9}{16}(t-4)^2 + 9\right)dt = 2.4375$

Total distance ≈ 53.4 E

SECTION 2

Q1a $y = \frac{9}{(x+2)(x-4)} = \frac{9}{x^2-2x-8}$, $\frac{1}{y} = \frac{x^2-2x-8}{9}$,

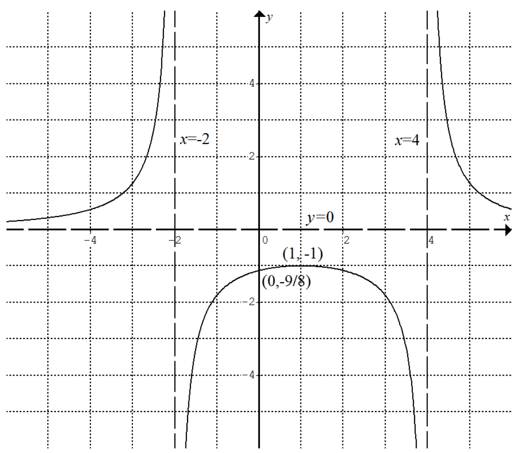
$-\frac{1}{y^2}\frac{dy}{dx} = \frac{2x-2}{9}$.

Let $\frac{dy}{dx} = 0$ . .: $x = 1$ and $y = -1$

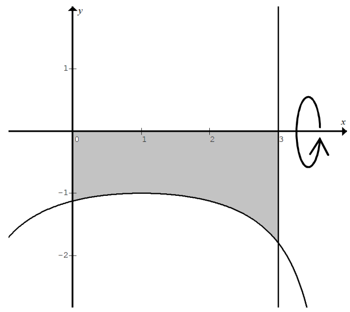
The stationary point is $(1,-1)$.

Q1b $x = -2$, $x = 4$, $y = 0$

Q1c



Q1di $V = \int_0^3 \pi\, y^2 dx = \int_0^3 \frac{81\pi}{(x+2)^2(x-4)^2}dx$



Q1dii By CAS, $V = 12.85$ cubic units

Q2ai $z_1 = \sqrt{3} - 3i$, $|z_1| = \sqrt{\left(\sqrt{3}\right)^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$

$\text{Arg}(z_1) = \tan^{-1}\left(\frac{-3}{\sqrt{3}}\right) = -\frac{\pi}{3}$, .: $z_1 = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$

Q2aii $z_1 = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$, $\arg\left(z_1^{\,4}\right) = 4\arg(z_1) = -\frac{4\pi}{3}$

.: $\text{Arg}\left(z_1^{\,4}\right) = \frac{2\pi}{3}$

Q2aiii $z_1 = \sqrt{3} - 3i$ is a root of $z^3 + 24\sqrt{3} = 0$, .: $z = \sqrt{3} + 3i$ is also a root.

.: $z^3 + 24\sqrt{3} = \left(z - \left(\sqrt{3} - 3i\right)\right)\left(z - \left(\sqrt{3} + 3i\right)\right)(z - p) = 0$ where $p \in R$

$z^3 + 24\sqrt{3} = \left(z^2 - 2\sqrt{3}\,z + 12\right)(z - p) = 0$

.: $-12p = 24\sqrt{3}$, .: $p = -2\sqrt{3}$

The other 2 roots are $\sqrt{3} + 3i$ and $-2\sqrt{3}$.

Q2bi $(z_1 + 2i)(\bar{z}_1 - 2i) = \left(\sqrt{3} - 3i + 2i\right)\left(\sqrt{3} + 3i - 2i\right)$

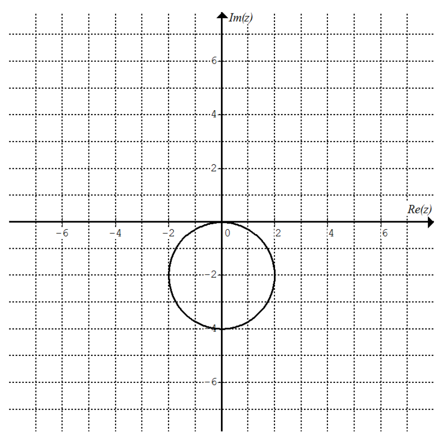
$= \left(\sqrt{3} - i\right)\left(\sqrt{3} + i\right) = 4$

Q2bii Let $z = x + yi$.

$(z + 2i)(z - 2i) = (x + (y+2)i)(x - (y+2)i) = 4$

.: $x^2 + (y+2)^2 = 4$

Q2biii



Q2c Let the line be $y = mx + c$. It passes through $(k, -2)$ and $(0, -(2+k))$, where $k < 0$.

$\therefore\ c = -(2+k)$ and $m = \dfrac{-2+(2+k)}{k-0} = 1$

$\therefore\ y = x-(2+k)$ is a tangent line to $x^2 + (y+2)^2 = 4$.

Solve the two equations simultaneously:

$x^2 + (x-(2+k)+2)^2 = 4$, $\therefore\ 2x^2 - 2kx + k^2 - 4 = 0$ and its discriminant $\Delta = 0$, i.e. $(-2k)^2 - 4(2)(k^2-4) = 0$

$\therefore\ -4k^2 + 32 = 0$, $\therefore\ k = -2\sqrt{2}$ since $k < 0$.

Q3a $\tilde{a} = 3\tilde{i} + 2\tilde{j} + \tilde{k}$, $\hat{b} = \dfrac{\tilde{b}}{|\tilde{b}|} = \dfrac{2\tilde{i} - 2\tilde{j} - \tilde{k}}{3}$, $\tilde{a}.\hat{b} = \dfrac{1}{3}$,

$\therefore$ the parallel vector resolute is $(\tilde{a}.\hat{b})\hat{b} = \dfrac{2}{9}\tilde{i} - \dfrac{2}{9}\tilde{j} - \dfrac{1}{9}\tilde{k}$,

and the perpendicular vector resolute is

$\tilde{a} - (\tilde{a}.\hat{b})\hat{b} = 3\tilde{i} + 2\tilde{j} + \tilde{k} - \left(\dfrac{2}{9}\tilde{i} - \dfrac{2}{9}\tilde{j} - \dfrac{1}{9}\tilde{k}\right) = \dfrac{25}{9}\tilde{i} + \dfrac{20}{9}\tilde{j} + \dfrac{10}{9}\tilde{k}$

$\therefore\ \tilde{a} = \left(\dfrac{2}{9}\tilde{i} - \dfrac{2}{9}\tilde{j} - \dfrac{1}{9}\tilde{k}\right) + \left(\dfrac{25}{9}\tilde{i} + \dfrac{20}{9}\tilde{j} + \dfrac{10}{9}\tilde{k}\right)$

$= \dfrac{1}{9}\left(2\tilde{i} - 2\tilde{j} - \tilde{k}\right) + \dfrac{5}{9}\left(5\tilde{i} + 4\tilde{j} + 2\tilde{k}\right)$

Q3bi $\overrightarrow{AP} = \alpha\overrightarrow{AD} = \alpha\left(\overrightarrow{AB} + \overrightarrow{BD}\right) = \alpha\left(\overrightarrow{OC} - \dfrac{1}{2}\overrightarrow{OA}\right) = \alpha\tilde{c} - \dfrac{1}{2}\alpha\tilde{a}$

Q3bii $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \beta\overrightarrow{OB} - \overrightarrow{OA} = \beta\left(\overrightarrow{OA} + \overrightarrow{OC}\right) - \overrightarrow{OA}$

$= \beta(\tilde{a} + \tilde{c}) - \tilde{a} = \beta\tilde{c} - (1-\beta)\tilde{a}$

Q3biii From parts i and ii, $\alpha\tilde{c} - \dfrac{1}{2}\alpha\tilde{a} = \beta\tilde{c} - (1-\beta)\tilde{a}$

$\therefore\ \alpha = \beta$ and $\dfrac{1}{2}\alpha = 1-\beta$, $\therefore\ \alpha = \beta = \dfrac{2}{3}$

Q4a $\dfrac{r}{h} = \dfrac{0.5}{1}$, $r = \dfrac{h}{2}$, $\therefore\ V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi}{12}h^3$

Q4b $\dfrac{dV}{dh} = \dfrac{\pi}{4}h^2$; $\dfrac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h} = 0.01\pi\left(2-\sqrt{h}\right)$

$\dfrac{dV}{dh} \times \dfrac{dh}{dt} = \dfrac{dV}{dt}$, $\dfrac{\pi}{4}h^2 \times \dfrac{dh}{dt} = 0.01\pi\left(2-\sqrt{h}\right)$, $\dfrac{dh}{dt} = \dfrac{0.04\left(2-\sqrt{h}\right)}{h^2}$

When $h = 0.25$, $\dfrac{dh}{dt} = 0.96$ m/min

Q4c $\dfrac{dh}{dt} = \dfrac{0.04\left(2-\sqrt{h}\right)}{h^2}$, $\dfrac{dt}{dh} = \dfrac{25h^2}{2-\sqrt{h}}$

$t = \displaystyle\int_0^1 \dfrac{25h^2}{2-\sqrt{h}}\,dh \approx 7.4$ minutes (By CAS)

Q4d $V = \dfrac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$, $\dfrac{dV}{dx} = \dfrac{\pi}{16}\left(x^2 + 4x + 4\right) = \dfrac{\pi}{16}(x+2)^2$

$\dfrac{dV}{dx} \times \dfrac{dx}{dt} = \dfrac{dV}{dt}$, $\dfrac{\pi}{16}(x+2)^2\dfrac{dx}{dt} = 0.05\pi$

$\therefore\ \dfrac{dx}{dt} = \dfrac{4}{5(x+2)^2}$, $\therefore\ \dfrac{dt}{dx} = \dfrac{5}{4}(x+2)^2$ and $x = 0$ at $t = 0$

$\therefore\ t = \displaystyle\int_0^x \dfrac{5}{4}(x+2)^2\,dx = \left[\dfrac{5(x+2)^3}{12}\right]_0^x = \dfrac{5(x+2)^3}{12} - \dfrac{10}{3}$

$\therefore\ \dfrac{12}{5}\left(t + \dfrac{10}{3}\right) = (x+2)^3$, $\therefore\ (x+2)^3 = 8(0.3t+1)$

$\therefore\ x = 2(0.3t+1)^{\frac{1}{3}} - 2$

Q5ai $T_1 - 2g = 2a$

Q5aii $T_2 + 5g\sin\theta - T_1 = 5a$; $3g\sin\theta - T_2 = 3a$

Q5aiii Add up the three equations:

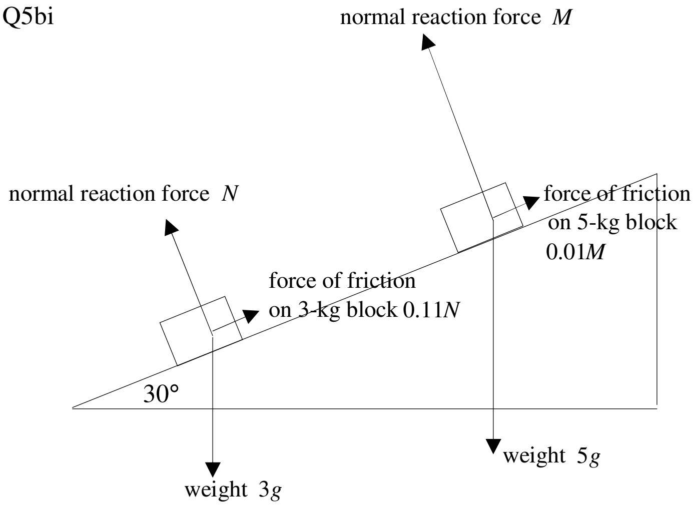
Sum of the left sides = sum of the right sides

$8g\sin\theta - 2g = 10a$, $\therefore\ a = \dfrac{g(4\sin\theta - 1)}{5}$

Q5aiv Net force is zero for the system to be in equilibrium,

$\therefore\ a = 0$, $\dfrac{g(4\sin\theta - 1)}{5} = 0$, $\sin\theta = \dfrac{1}{4}$, $\theta \approx 14.5°$

Q5bi



Q5bii 3 kg: $3g\sin 30° - 0.11 \times 3g\cos 30° = 3a$, $a \approx 3.97$ m/s^2

5 kg: $5g\sin 30° - 0.01 \times 5g\cos 30° = 5a$, $a \approx 4.82$ m/s^2

Q5biii Both blocks start from rest. The 5 kg block moves 3 extra metres when it collides with the 3 kg block at time t seconds.

$\therefore\ \dfrac{1}{2} \times 3.97t^2 + 3 = \dfrac{1}{2} \times 4.82t^2$, $\therefore\ t \approx 2.66$ s

Please inform mathline@itute.com re conceptual and/or mathematical errors

