2017 NSW ESA Mathematics Extension 2 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
C	B	D	C	B	A	A	B	C	B

Q1 There are 8 solutions, .: polynomial equation is of degree 8, and 1 is one of the solutions as shown in the given diagram. **C**

Q2 **B**

Q3 $|(3-i)-1| = |2-i| = \sqrt{5}$, $2 < \sqrt{5} < 3$ **D**

Q4 A and B are undefined for $x < -1$ or $0 < x < 2$.
D is an even function. **C**

Q5 $\alpha^3 = 2\alpha - 2$, $\beta^3 = 2\beta - 2$, $\gamma^3 = 2\gamma - 2$

.: $\alpha^3 + \beta^3 + \gamma^3 = 2(\alpha + \beta + \gamma) - 6 = 2(0) - 6 = -6$ **B**

Q6 $\alpha\beta\gamma = -15$, $\alpha(2+i)(2-i) = -15$, $\alpha = -3$

.: $-3 + (2+i) + (2-i) = -a$, .: $a = -1$ **A**

Q7 $\int_{-a}^{a} (f(x) + g(x))\,dx = \int_{-a}^{a} f(x)\,dx + \int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} f(x)\,dx + 0$ **A**

Q8 Since $f(x)$ is an odd function,

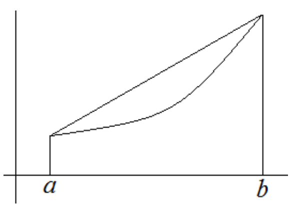
.: $f(f(-x)) = f(-f(x)) = -f(f(x))$, .: $f(f(x))$ is odd. **B**

Q9 $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$, $\frac{dy}{dt} = -\frac{x}{y} \times \frac{dx}{dt} = -\frac{x}{y} \times y = -x$

.: for positive x, y decreases at a higher rate as x increases

.: particle travels clockwise **C**

Q10 A possible graph of $f(x)$ is shown below.



When $a = 0$ and $b = 1$, area under the curve $\int_{0}^{1} f(x)\,dx$ is less than the area of the trapezium $\frac{f(0) + f(1)}{2}$. **B**

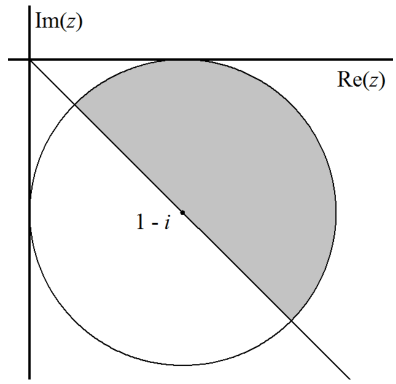
Section II

Q11ai $z = 1 - \sqrt{3}i$, $\arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

Q11aii $w = 1 + i$, $\arg(w) = \frac{\pi}{4}$, $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) = -\frac{7\pi}{12}$

Q11b $\tan\alpha = \frac{2}{\sqrt{12}} = \frac{1}{\sqrt{3}}$, .: $\alpha = \frac{\pi}{6}$

Q11c Shaded region

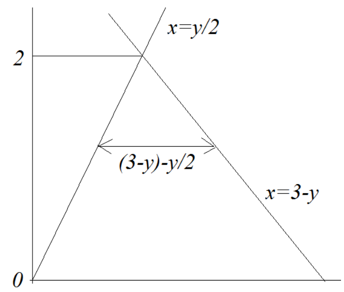


Q11d $t = \tan\frac{\theta}{2}$, $\theta = 2\tan^{-1} t$, $\frac{d\theta}{dt} = \frac{2}{1+t^2}$

$1 + \cos\theta = 1 + \frac{1-t^2}{1+t^2} = \frac{2}{1+t^2}$

$\int_{0}^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\,d\theta = \int_{0}^{\frac{2\pi}{3}} \frac{1}{\frac{d\theta}{dt}}\,d\theta = \int_{0}^{\frac{2\pi}{3}} \frac{dt}{d\theta}\,d\theta = \int_{0}^{\sqrt{3}} dt = \sqrt{3}$

Q11e



$V = \int_{0}^{2} 2\pi y\left(3 - y - \frac{y}{2}\right)dy = \int_{0}^{2} 3\pi(2y - y^2)\,dy$

Q11f $x = \sin^2\theta$, $\theta = \sin^{-1}\sqrt{x}$

$\frac{d\theta}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{\sqrt{x}}{2x\sqrt{1-x}} = \frac{1}{2x}\sqrt{\frac{x}{1-x}}$, .: $\sqrt{\frac{x}{1-x}} = 2x\frac{d\theta}{dx}$

$\int_{0}^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx = \int_{0}^{\frac{1}{2}} 2x\frac{d\theta}{dx}\,dx = \int_{0}^{\frac{\pi}{4}} 2\sin^2\theta\,d\theta = \int_{0}^{\frac{\pi}{4}} (1 - \cos 2\theta)\,d\theta$

$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}$

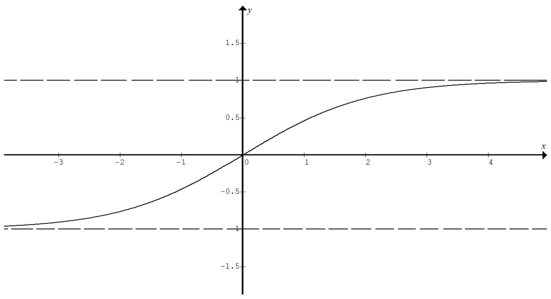
Q12ai $f(x)=\frac{e^x-1}{e^x+1}=1-\frac{2}{e^x+1}$, $f'(x)=\frac{2e^x}{(e^x+1)^2}>0$ for $x\in R$

.: $f(x)$ is increasing for all real x.

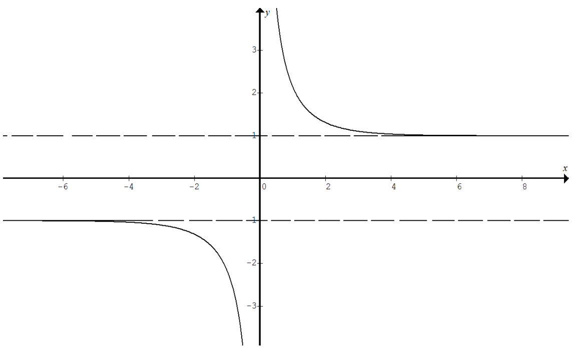
Q12aii $f(-x)=\frac{e^{-x}-1}{e^{-x}+1}=\frac{1-e^x}{1+e^x}=-f(x)$ for $x\in R$, .: $f(x)$ is odd.

Q12aiii $f(x)=\frac{e^x-1}{e^x+1}=1-\frac{2}{e^x+1}$. As $x\to+\infty$, $f(x)\to 1^-$.

Q12aiv



Q12av



Q12b $z^2+(2+3i)z+(1+3i)=z^2+(1+(1+3i))z+1(1+3i)$

$=(z+1)(z+(1+3i))=0$, .: $z=-1$ or $-1-3i$

Q12c Let $u=\tan^{-1}x$ and $\frac{dv}{dx}=x$, .: $\frac{du}{dx}=\frac{1}{1+x^2}$, $v=\frac{x^2}{2}$.

$\int x\tan^{-1}x\,dx=\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\times\frac{1}{1+x^2}dx$

$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$

$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+c$

$=\frac{x^2+1}{2}\tan^{-1}x-\frac{x}{2}+c$

Q12di $P(x)=(x-\alpha)^2Q(x)$, $P'(x)=2(x-\alpha)Q(x)+(x-\alpha)^2Q'(x)$

.: $P(\alpha)=0$ and $P'(\alpha)=0$

Q12dii $P(x)=x^4-3x^3+x^2+4$, $P'(x)=4x^3-9x^2+2x$

From part i,

$P(\alpha)=\alpha^4-3\alpha^3+\alpha^2+4=0$ …….(I), and

$P'(\alpha)=4\alpha^3-9\alpha^2+2\alpha=0$ ………(II)

$P'(\alpha)=\alpha(4\alpha-1)(\alpha-2)=0$

Only $\alpha=2$ satisfies both I and II.

Q13a $\left(\sqrt{r}-\sqrt{s}\right)^2\ge 0$, $r-2\sqrt{r}\sqrt{s}+s\ge 0$, $r+s\ge 2\sqrt{rs}$

.: $\frac{r+s}{2}\ge\sqrt{rs}$

Q13bi $\alpha+\frac{1}{\alpha}+\beta+\frac{1}{\beta}=-a$

$\alpha\left(\frac{1}{\alpha}\right)\beta+\left(\frac{1}{\alpha}\right)\beta\left(\frac{1}{\beta}\right)+\beta\left(\frac{1}{\beta}\right)\alpha+\left(\frac{1}{\beta}\right)\alpha\left(\frac{1}{\alpha}\right)=-c$

.: $\alpha+\frac{1}{\alpha}+\beta+\frac{1}{\beta}=-c$, .: $a=c$

Q13bii $b=\alpha\left(\frac{1}{\alpha}\right)+\left(\frac{1}{\alpha}\right)\beta+\beta\left(\frac{1}{\beta}\right)+\left(\frac{1}{\beta}\right)\alpha+\alpha\beta+\frac{1}{\alpha\beta}$

$=2+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\alpha\beta+\frac{1}{\alpha\beta}$

$\ge 2+2\sqrt{\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}}+2\sqrt{\alpha\beta\times\frac{1}{\alpha\beta}}$, .: $b\ge 6$

Q13c $\ddot{x}=-(g+kv^2)$, $\frac{1}{2}\frac{d(v^2)}{dx}=-(g+kv^2)$, $2\frac{dx}{d(v^2)}=-\frac{1}{g+kv^2}$,

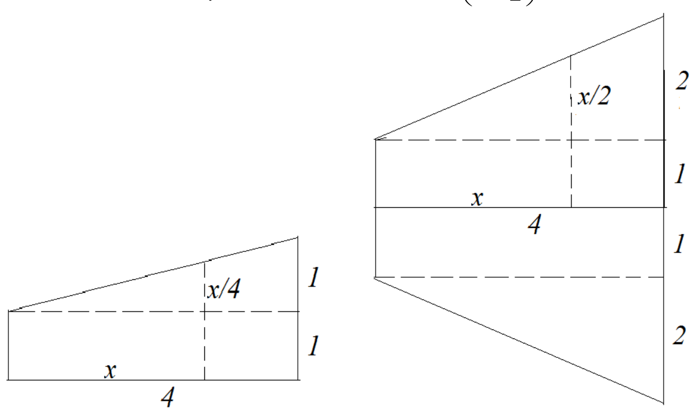
$2x=-\int\frac{1}{g+kv^2}d(v^2)=-\frac{1}{k}\log_e(g+kv^2)+c$

At ground level, $x=0$ and $v=\frac{1}{2}\sqrt{\frac{g}{k}}$, .: $c=\frac{1}{k}\log_e\frac{5g}{4}$

.: $2x=\frac{1}{k}\log_e\left(\frac{\frac{5g}{4}}{g+kv^2}\right)$

When $x=H$, $v=0$, .: $H=\frac{1}{2k}\log_e\left(\frac{5}{4}\right)$

Q13d Height $=1+\frac{x}{4}$ Base $=2\left(1+\frac{x}{2}\right)=2+x$



Area of $\Delta PQR=\frac{1}{2}(2+x)\left(1+\frac{x}{4}\right)=\frac{1}{8}(x^2+6x+8)$

Volume $=\int_0^4\frac{1}{8}(x^2+6x+8)dx=\frac{1}{8}\left[\frac{x^3}{3}+3x^2+8x\right]_0^4=\frac{38}{3}$

Q13e $\overrightarrow{AC}=c-a$, $\overrightarrow{AD}=d-a$, $\overrightarrow{DC}=c-d$

$\overrightarrow{DC}$ is the anticlockwise rotation of $\overrightarrow{AD}$ by 90°, .: $\overrightarrow{DC}=i(d-a)$

Since $\overrightarrow{AC}=\overrightarrow{AD}+\overrightarrow{DC}$, .: $c-a=d-a+i(d-a)$, .: $c=(1+i)d-ia$

Q14ai $(A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)=16$ for all x

Let $x=0$, $A+B=8$. Let $x=1$, $A+5B=24$. $\therefore$ $A=B=4$

Q14aii $\int_0^m \frac{16}{x^4+4}dx=\int_0^m \frac{2x+4}{x^2+2x+2}dx-\int_0^m \frac{2x-4}{x^2-2x+2}dx$

$=\int_0^m \frac{2x+2}{x^2+2x+2}dx+\int_0^m \frac{2}{x^2+2x+2}dx-\int_0^m \frac{2x-2}{x^2-2x+2}dx+\int_0^m \frac{2}{x^2-2x+2}dx$

$=\int_0^m \frac{2x+2}{x^2+2x+2}dx+\int_0^m \frac{2}{(x+1)^2+1}dx-\int_0^m \frac{2x-2}{x^2-2x+2}dx+\int_0^m \frac{2}{(x-1)^2+1}dx$

$=\left[\log_e\left(\frac{x^2+2x+2}{x^2-2x+2}\right)+2\tan^{-1}(x+1)+2\tan^{-1}(x-1)\right]_0^m$

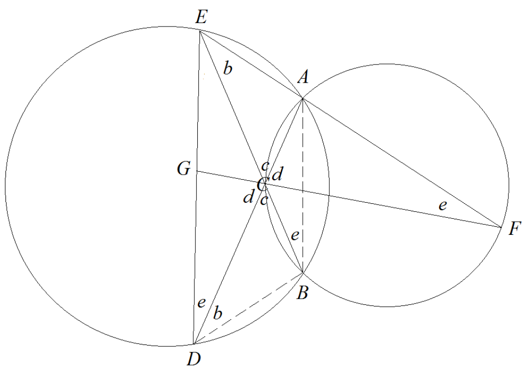
$=\log_e\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)$

Q14aiii $\lim_{m\to\infty}\int_0^m \frac{16}{x^4+4}dx$

$=\lim_{m\to\infty}\left[\log_e\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)\right]$

$=\log_e 1+\pi+\pi=2\pi$

Q14b Two dotted lines are added to the diagram to define angles. b, c, d and e are used to label equal angles.



Q14bi $\angle EAD=\angle EBD$, because the same chord *ED* subtends the two angles on the circumference of circle 1.

Q14bii $\angle EDA=\angle AFC=\angle ABC=e$. See diagram above.

Q14biii In ΔCEF, $b+(c+d)+e=180°$

$\therefore$ $(c+d)+(b+e)=180°$ in quadrilateral *BCGD*.

$\therefore$ the two opposite angles in quadrilateral *BCGD* are supplementary. Hence *B*, *C*, *G* and *D* are concyclic points.

Q14ci Vertical: $R\sin\theta-mg=0$ Horizontal: $R\cos\theta=mr\omega^2$

Solving simultaneously, $\omega^2=\frac{g}{r\tan\theta}=\frac{g}{r\left(\frac{r}{h}\right)}=\frac{gh}{r^2}$

Q14cii Vertical: $N\sin\theta+T\cos\theta-mg=0$

Horizontal: $T\sin\theta-N\cos\theta=mr\omega^2$ where $\omega^2=\frac{gh}{r^2}$

Eliminating T, $N=mg\sin\theta-\frac{mgh}{r}\cos\theta=mg\left(\sin\theta-\frac{h}{r}\cos\theta\right)$

Q14cii Since $N\ge 0$, $\sin\theta-\frac{h}{r}\cos\theta\ge 0$, $\therefore$ $\tan\theta\ge\frac{h}{r}$

$\therefore$ $\tan\theta\ge\frac{1}{\tan\theta}$, $\tan^2\theta\ge 1$, $\tan\theta\ge 1$, $\therefore$ $\theta\ge\frac{\pi}{4}$

Q15ai $I_1=\int_0^1 x\sqrt{1-x^2}\,dx=\left[-\frac{\left(\sqrt{1-x^2}\right)^3}{3}\right]_0^1=\frac{1}{3}$

Q15aii $I_n=\int_0^1 x^n\sqrt{1-x^2}\,dx=\int_0^1 x^{n-1}x\sqrt{1-x^2}\,dx$

Let $u=x^{n-1}$ and $\frac{dv}{dx}=x\sqrt{1-x^2}$,

then $\frac{du}{dx}=(n-1)x^{n-2}$, $v=-\frac{\left(\sqrt{1-x^2}\right)^3}{3}$.

$I_n=[uv]_0^1-\int_0^1 v\frac{du}{dx}dx=\left[-\frac{x^{n-1}\left(\sqrt{1-x^2}\right)^3}{3}\right]_0^1-\int_0^1 -\frac{(n-1)x^{n-2}\left(\sqrt{1-x^2}\right)^3}{3}dx$

$=\frac{n-1}{3}\int_0^1 x^{n-2}\left(\sqrt{1-x^2}\right)^3dx=\frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$

$=\frac{n-1}{3}\left(\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx-\int_0^1 x^n\sqrt{1-x^2}\,dx\right)=\frac{n-1}{3}(I_{n-2}-I_n)$

$I_n=\frac{n-1}{3}(I_{n-2}-I_n)$, $\therefore$ $I_n=\left(\frac{n-1}{n+2}\right)I_{n-2}$

Q15aiii $I_5=\left(\frac{5-1}{5+2}\right)I_3=\left(\frac{4}{7}\right)\left(\frac{3-1}{3+2}\right)I_1=\left(\frac{4}{7}\right)\left(\frac{2}{5}\right)\left(\frac{1}{3}\right)=\frac{8}{105}$

Q15bi $\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}\sqrt{a}$, $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$

$\therefore$ $\frac{dy}{dx}=-\sqrt{\frac{y}{x}}=-\sqrt{\frac{d}{c}}$ at $P(c,d)$

Equation of the tangent at $P(c,d)$:

$y-d=-\sqrt{\frac{d}{c}}(x-c)$, $\therefore$ $y\sqrt{c}+x\sqrt{d}=d\sqrt{c}+c\sqrt{d}$

Q15bii Let $y=0$, $x=\sqrt{cd}+c$; let $x=0$, $y=\sqrt{cd}+d$

$OA+OB=x+y=c+2\sqrt{cd}+d=\left(\sqrt{c}+\sqrt{d}\right)^2=\left(\sqrt{a}\right)^2=a$

Q15ci $P(x_1, y_1)$ is on both conics.

$\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\frac{x_1^2}{c^2}-\frac{y_1^2}{d^2}=1$

$\therefore$ $\frac{y_1^2}{d^2}+\frac{y_1^2}{b^2}=\frac{x_1^2}{c^2}-\frac{x_1^2}{a^2}$

$\therefore$ $\left(\frac{1}{d^2}+\frac{1}{b^2}\right)y_1^2=\left(\frac{1}{c^2}-\frac{1}{a^2}\right)x_1^2$

$\therefore$ $\frac{x_1^2}{y_1^2}=\frac{a^2c^2}{(a^2-c^2)}\times\frac{(b^2+d^2)}{b^2d^2}$

Q15cii Ellipse: $b^2 = a^2\left(1-e^2\right)$, .: $ae = \sqrt{a^2 - b^2}$

$$m_e = \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y} = -\frac{b^2}{a^2}\frac{x_1}{y_1} \text{ at } P\left(x_1,\, y_1\right)$$

Hyperbola: $d^2 = c^2\left(E^2 - 1\right)$, .: $cE = \sqrt{c^2 + d^2}$

$$m_h = \frac{dy}{dx} = \frac{d^2}{c^2}\frac{x}{y} = \frac{d^2}{c^2}\frac{x_1}{y_1} \text{ at } P\left(x_1,\, y_1\right)$$

Both conics have the same foci, .: $ae = cE$, $\sqrt{a^2 - b^2} = \sqrt{c^2 + d^2}$

.: $a^2 - b^2 = c^2 + d^2$, $a^2 - c^2 = b^2 + d^2$

$$.:\ \frac{{x_1}^2}{{y_1}^2} = \frac{a^2c^2}{\left(a^2 - c^2\right)} \times \frac{\left(b^2 + d^2\right)}{b^2d^2} = \frac{a^2c^2}{b^2d^2}$$

.: the product of the two gradients $= m_e m_h = -\frac{b^2d^2}{a^2c^2}\frac{{x_1}^2}{{y_1}^2} = -1$

.: the tangents to the two conics at $P\left(x_1,\, y_1\right)$ are perpendicular.

Q16ai $\alpha = \cos\theta + i\sin\theta$

$$\alpha^k + \alpha^{-k} = \cos k\theta + i\sin k\theta + \cos(-k\theta) + i\sin(-k\theta)$$

$$= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta$$

$$= 2\cos k\theta$$

Q16aii Geometric series: $a = \alpha^n$, $r = \alpha^{-1}$, number of terms $= 2n+1$

$$C = \frac{\alpha^n\left(1 - \left(\alpha^{-1}\right)^{2n+1}\right)}{1 - \alpha^{-1}} = \frac{\alpha^n\left(1 - \alpha^{-2n-1}\right)}{1 - \alpha^{-1}}$$

$$= \frac{(1-\alpha)\left(\alpha^n - \alpha^{-n-1}\right)}{(1-\alpha)\left(1-\alpha^{-1}\right)} = \frac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\overline{\alpha})}$$

Q16aiii $\alpha^{-n} + ... + \alpha^{-1} + 1 + \alpha + \alpha^n = \frac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\overline{\alpha})}$

$$1 + \left(\alpha + \alpha^{-1}\right) + \left(\alpha^2 + \alpha^{-2}\right) ... + \left(\alpha^n + \alpha^{-n}\right) = \frac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{2 - \left(\alpha + \alpha^{-1}\right)}$$

$$1 + 2\cos\theta + 2\cos 2\theta + ... + 2\cos n\theta = \frac{2\cos n\theta - 2\cos(n+1)\theta}{2 - 2\cos\theta}$$

$$.:\ 1 + 2(\cos\theta + \cos 2\theta + ... + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$

Q16aiv Let $\theta = \frac{\pi}{n}$.

$$.:\ 1 + 2\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + ... + \cos\frac{n\pi}{n}\right) = \frac{\cos\frac{n\pi}{n} - \cos\frac{(n+1)\pi}{n}}{1 - \cos\frac{\pi}{n}}$$

$$.:\ 1 + 2\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + ... + \cos\frac{n\pi}{n}\right) = \frac{-1 - \cos\left(\pi + \frac{\pi}{n}\right)}{1 - \cos\frac{\pi}{n}}$$

$$.:\ 1 + 2\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + ... + \cos\frac{n\pi}{n}\right) = \frac{-1 + \cos\frac{\pi}{n}}{1 - \cos\frac{\pi}{n}} = -1$$

.: $\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + ... + \cos\frac{n\pi}{n}$ is independent of n.

Q16bi $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a, b > 0$

Vertices: $x = \pm a$; foci: $x = \pm 2a$

$2a - a = 1$ OR $2a + a = 1$

.: $a = 1$ OR $a = \frac{1}{3}$

Q16ci

$$N(\text{colours for the second col}) = N(\text{C same as B}) + N(\text{C diff. from B})$$

$$= N_s(\text{C}) \times N_s(\text{D}) + N_d(\text{C}) \times N_d(\text{D})$$

$$= 1(x-1) + (x-1)(x-1)$$

$$= x^2 - 3x + 3$$

Q16cii For 1 column (2 by 1), it is given $N = x(x-1)$

$N = x(x-1)\left(x^2 - 3x + 3\right)^{1-1} = x(x-1)$ for $n = 1$

.: true for $n = 1$

Assuming $N = x(x-1)\left(x^2 - 3x + 3\right)^{n-1}$ is true for n columns.

For the next column, i.e. the $(n+1)$th column, the number of ways is $x^2 - 3x + 3$.

.: the total number of ways to paint the 2 by $n+1$ grid

$$= x(x-1)\left(x^2 - 3x + 3\right)^{n-1} \times \left(x^2 - 3x + 3\right)$$

$$= x(x-1)\left(x^2 - 3x + 3\right)^n$$

.: true for $n \geq 1$

Q16ciii The total number of ways to paint the 2 by 5 grid using all 3 colours at least once

$$= N(n = 5, x = 3) - {}^3C_2 \times N(n = 5, x = 2)$$

$$= 3(3-1)\left(3^2 - 3\times 3 + 3\right)^{5-1} - 3\times 2(2-1)\left(2^2 - 3\times 2 + 3\right)^{5-1}$$

$$= 480$$

Please inform mathline@itute.com re conceptual and/or mathematical errors.