



2017 NSW ESA Mathematics Extension 2 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
C	B	D	C	B	A	A	B	C	B

Q1 There are 8 solutions, \therefore polynomial equation is of degree 8, and 1 is one of the solutions as shown in the given diagram. **C**

Q2 **B**

Q3 $|(3-i)-1| = |2-i| = \sqrt{5}$, $2 < \sqrt{5} < 3$ **D**

Q4 A and B are undefined for $x < -1$ or $0 < x < 2$.
D is an even function. **C**

Q5 $\alpha^3 = 2\alpha - 2$, $\beta^3 = 2\beta - 2$, $\gamma^3 = 2\gamma - 2$
 $\therefore \alpha^3 + \beta^3 + \gamma^3 = 2(\alpha + \beta + \gamma) - 6 = 2(0) - 6 = -6$ **B**

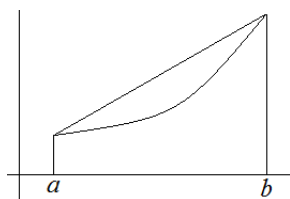
Q6 $\alpha\beta\gamma = -15$, $\alpha(2+i)(2-i) = -15$, $\alpha = -3$
 $\therefore -3 + (2+i) + (2-i) = -a$, $\therefore a = -1$ **A**

Q7 $\int_{-a}^a (f(x) + g(x)) dx = \int_{-a}^a f(x) dx + \int_{-a}^a g(x) dx = 2 \int_0^a f(x) dx + 0$ **A**

Q8 Since $f(x)$ is an odd function,
 $\therefore f(f(-x)) = f(-f(x)) = -f(f(x))$, $\therefore f(f(x))$ is odd. **B**

Q9 $x \frac{dx}{dt} + y \frac{dy}{dt} = 0$, $\frac{dy}{dt} = -\frac{x}{y} \times \frac{dx}{dt} = -\frac{x}{y} \times y = -x$
 \therefore for positive x , y decreases at a higher rate as x increases
 \therefore particle travels clockwise **C**

Q10 A possible graph of $f(x)$ is shown below.



When $a = 0$ and $b = 1$, area under the curve $\int_0^1 f(x) dx$ is less than the area of the trapezium $\frac{f(0) + f(1)}{2}$. **B**

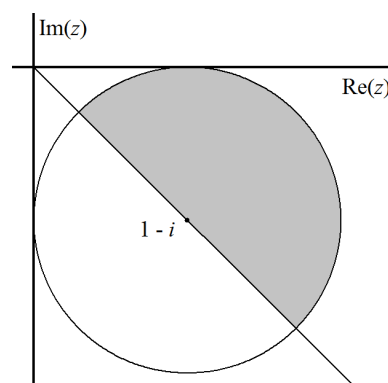
Section II

Q11ai $z = 1 - \sqrt{3}i$, $\arg(z) = \frac{-\sqrt{3}}{1} = -\frac{\pi}{3}$

Q11aai $w = 1 + i$, $\arg(w) = \frac{\pi}{4}$, $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) = -\frac{7\pi}{12}$

Q11b $\tan \alpha = \frac{2}{\sqrt{12}} = \frac{1}{\sqrt{3}}$, $\therefore \alpha = \frac{\pi}{6}$

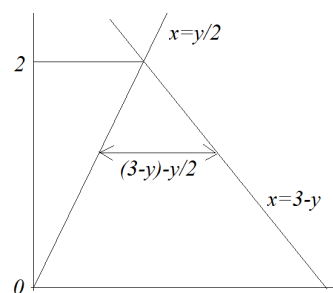
Q11c Shaded region



Q11d $t = \tan \frac{\theta}{2}$, $\theta = 2 \tan^{-1} t$, $\frac{d\theta}{dt} = \frac{2}{1+t^2}$

$1 + \cos \theta = 1 + \frac{1-t^2}{1+t^2} = \frac{2}{1+t^2}$
 $\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos \theta} d\theta = \int_0^{\frac{2\pi}{3}} \frac{1}{\frac{2}{1+t^2}} d\theta = \int_0^{\frac{2\pi}{3}} \frac{1+t^2}{2} d\theta = \int_0^{\frac{2\pi}{3}} \frac{dt}{d\theta} d\theta = \int_0^{\sqrt{3}} dt = \sqrt{3}$

Q11e



$V = \int_0^2 2\pi y \left(3 - y - \frac{y}{2}\right) dy = \int_0^2 3\pi(2y - y^2) dy$

Q11f $x = \sin^2 \theta$, $\theta = \sin^{-1} \sqrt{x}$

$\frac{d\theta}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{\sqrt{x}}{2x\sqrt{1-x}} = \frac{1}{2x\sqrt{1-x}}$, $\therefore \sqrt{\frac{x}{1-x}} = 2x \frac{d\theta}{dx}$

$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}} dx = \int_0^{\frac{1}{2}} 2x \frac{d\theta}{dx} dx = \int_0^{\frac{\pi}{4}} 2 \sin^2 \theta d\theta = \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta) d\theta$

$= \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}$



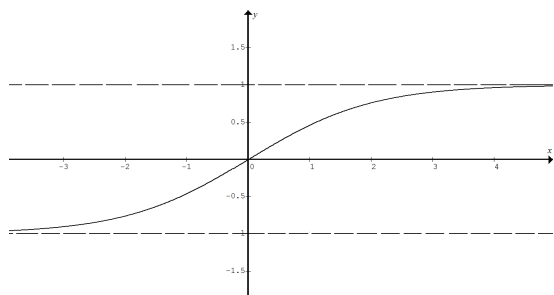
Q12ai $f(x) = \frac{e^x - 1}{e^x + 1} = 1 - \frac{2}{e^x + 1}$, $f'(x) = \frac{2e^x}{(e^x + 1)^2} > 0$ for $x \in \mathbb{R}$

$\therefore f(x)$ is increasing for all real x .

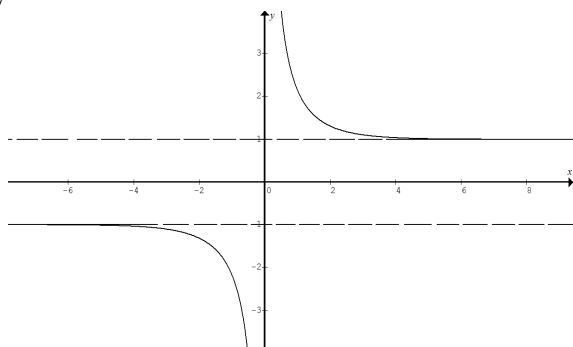
Q12aii $f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} = \frac{1 - e^x}{1 + e^x} = -f(x)$ for $x \in \mathbb{R}$, $\therefore f(x)$ is odd.

Q12aiii $f(x) = \frac{e^x - 1}{e^x + 1} = 1 - \frac{2}{e^x + 1}$. As $x \rightarrow +\infty$, $f(x) \rightarrow 1^-$.

Q12aiv



Q12av



Q12b $z^2 + (2 + 3i)z + (1 + 3i) = z^2 + (1 + (1 + 3i))z + 1(1 + 3i)$
 $= (z + 1)(z + (1 + 3i)) = 0$, $\therefore z = -1$ or $-1 - 3i$

Q12c Let $u = \tan^{-1} x$ and $\frac{dv}{dx} = x$, $\therefore \frac{du}{dx} = \frac{1}{1 + x^2}$, $v = \frac{x^2}{2}$.

$$\int x \tan^{-1} x \, dx = \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx = \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \times \frac{1}{1 + x^2} \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1 + x^2}\right) \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + c$$

$$= \frac{x^2 + 1}{2} \tan^{-1} x - \frac{x}{2} + c$$

Q12di $P(x) = (x - \alpha)^2 Q(x)$, $P'(x) = 2(x - \alpha)Q(x) + (x - \alpha)^2 Q'(x)$
 $\therefore P(\alpha) = 0$ and $P'(\alpha) = 0$

Q12dii $P(x) = x^4 - 3x^3 + x^2 + 4$, $P'(x) = 4x^3 - 9x^2 + 2x$

From part i,

$P(\alpha) = \alpha^4 - 3\alpha^3 + \alpha^2 + 4 = 0$ (I), and

$P'(\alpha) = 4\alpha^3 - 9\alpha^2 + 2\alpha = 0$ (II)

$P'(\alpha) = \alpha(4\alpha - 1)(\alpha - 2) = 0$

Only $\alpha = 2$ satisfies both I and II.

Q13a $(\sqrt{r} - \sqrt{s})^2 \geq 0$, $r - 2\sqrt{r}\sqrt{s} + s \geq 0$, $r + s \geq 2\sqrt{rs}$
 $\therefore \frac{r+s}{2} \geq \sqrt{rs}$

Q13bi $\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -a$

$\alpha\left(\frac{1}{\alpha}\right)\beta + \left(\frac{1}{\alpha}\right)\beta\left(\frac{1}{\beta}\right) + \beta\left(\frac{1}{\beta}\right)\alpha + \left(\frac{1}{\beta}\right)\alpha\left(\frac{1}{\alpha}\right) = -c$

$\therefore \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -c$, $\therefore a = c$

Q13bii $b = \alpha\left(\frac{1}{\alpha}\right) + \left(\frac{1}{\alpha}\right)\beta + \beta\left(\frac{1}{\beta}\right) + \left(\frac{1}{\beta}\right)\alpha + \alpha\beta + \frac{1}{\alpha\beta}$

$= 2 + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \alpha\beta + \frac{1}{\alpha\beta}$

$\geq 2 + 2\sqrt{\frac{\alpha}{\beta} \times \frac{\beta}{\alpha}} + 2\sqrt{\alpha\beta \times \frac{1}{\alpha\beta}}$, $\therefore b \geq 6$

Q13c $\ddot{x} = -(g + kv^2)$, $\frac{1}{2} \frac{d(v^2)}{dx} = -(g + kv^2)$, $2 \frac{dx}{d(v^2)} = -\frac{1}{g + kv^2}$,

$2x = -\int \frac{1}{g + kv^2} d(v^2) = -\frac{1}{k} \log_e(g + kv^2) + c$

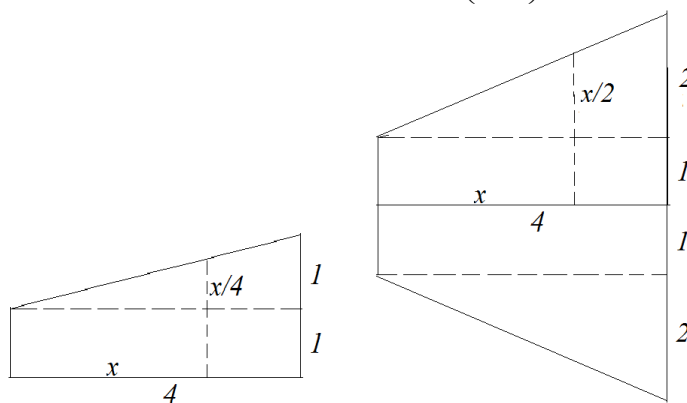
At ground level, $x = 0$ and $v = \frac{1}{2} \sqrt{\frac{g}{k}}$, $\therefore c = \frac{1}{k} \log_e \frac{5g}{4}$

$\therefore 2x = \frac{1}{k} \log_e \left(\frac{5g}{4} \right)$

When $x = H$, $v = 0$, $\therefore H = \frac{1}{2k} \log_e \left(\frac{5}{4} \right)$

Q13d Height = $1 + \frac{x}{4}$

Base = $2 \left(1 + \frac{x}{2}\right) = 2 + x$



Area of $\Delta PQR = \frac{1}{2}(2 + x) \left(1 + \frac{x}{4}\right) = \frac{1}{8}(x^2 + 6x + 8)$

Volume = $\int_0^4 \frac{1}{8}(x^2 + 6x + 8) \, dx = \frac{1}{8} \left[\frac{x^3}{3} + 3x^2 + 8x \right]_0^4 = \frac{38}{3}$

Q13e $\overrightarrow{AC} = c - a$, $\overrightarrow{AD} = d - a$, $\overrightarrow{DC} = c - d$

\overrightarrow{DC} is the anticlockwise rotation of \overrightarrow{AD} by 90° , $\therefore \overrightarrow{DC} = i(d - a)$

Since $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$, $\therefore c - a = d - a + i(d - a)$, $\therefore c = (1 + i)d - ia$

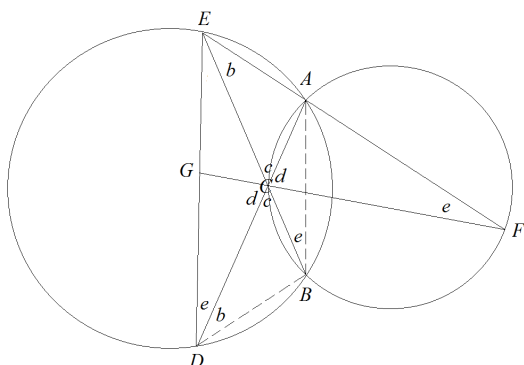


Q14ai $(A+2x)(x^2-2x+2)+(B-2x)(x^2+2x+2)=16$ for all x
 Let $x=0$, $A+B=8$. Let $x=1$, $A+5B=24$. $\therefore A=B=4$

$$\begin{aligned} \text{Q14aii } \int_0^m \frac{16}{x^4+4} dx &= \int_0^m \frac{2x+4}{x^2+2x+2} dx - \int_0^m \frac{2x-4}{x^2-2x+2} dx \\ &= \int_0^m \frac{2x+2}{x^2+2x+2} dx + \int_0^m \frac{2}{x^2+2x+2} dx - \int_0^m \frac{2x-2}{x^2-2x+2} dx + \int_0^m \frac{2}{x^2-2x+2} dx \\ &= \int_0^m \frac{2x+2}{x^2+2x+2} dx + \int_0^m \frac{2}{(x+1)^2+1} dx - \int_0^m \frac{2x-2}{x^2-2x+2} dx + \int_0^m \frac{2}{(x-1)^2+1} dx \\ &= \left[\log_e \left(\frac{x^2+2x+2}{x^2-2x+2} \right) + 2 \tan^{-1}(x+1) + 2 \tan^{-1}(x-1) \right]_0^m \\ &= \log_e \left(\frac{m^2+2m+2}{m^2-2m+2} \right) + 2 \tan^{-1}(m+1) + 2 \tan^{-1}(m-1) \end{aligned}$$

$$\begin{aligned} \text{Q14aiii } \lim_{m \rightarrow \infty} \int_0^m \frac{16}{x^4+4} dx &= \lim_{m \rightarrow \infty} \left[\log_e \left(\frac{m^2+2m+2}{m^2-2m+2} \right) + 2 \tan^{-1}(m+1) + 2 \tan^{-1}(m-1) \right] \\ &= \log_e 1 + \pi + \pi = 2\pi \end{aligned}$$

Q14b Two dotted lines are added to the diagram to define angles b , c , d and e are used to label equal angles.



Q14bi $\angle EAD = \angle EBD$, because the same chord ED subtends the two angles on the circumference of circle 1.

Q14bii $\angle EDA = \angle AFC = \angle ABC = e$. See diagram above.

Q14biii In $\triangle CEF$, $b+(c+d)+e=180^\circ$

$\therefore (c+d)+(b+e)=180^\circ$ in quadrilateral $BCGD$.

\therefore the two opposite angles in quadrilateral $BCGD$ are supplementary. Hence B, C, G and D are concyclic points.

Q14ci Vertical: $R \sin \theta - mg = 0$ Horizontal: $R \cos \theta = mr\omega^2$

$$\text{Solving simultaneously, } \omega^2 = \frac{g}{r \tan \theta} = \frac{g}{r \left(\frac{r}{h} \right)} = \frac{gh}{r^2}$$

Q14cii Vertical: $N \sin \theta + T \cos \theta - mg = 0$

Horizontal: $T \sin \theta - N \cos \theta = mr\omega^2$ where $\omega^2 = \frac{gh}{r^2}$

$$\text{Eliminating } T, N = mg \sin \theta - \frac{mgh}{r} \cos \theta = mg \left(\sin \theta - \frac{h}{r} \cos \theta \right)$$

Q14cii Since $N \geq 0$, $\sin \theta - \frac{h}{r} \cos \theta \geq 0$, $\therefore \tan \theta \geq \frac{h}{r}$

$$\therefore \tan \theta \geq \frac{1}{\tan \theta}, \tan^2 \theta \geq 1, \tan \theta \geq 1, \therefore \theta \geq \frac{\pi}{4}$$

$$\text{Q15ai } I_1 = \int_0^1 x\sqrt{1-x^2} dx = \left[-\frac{(\sqrt{1-x^2})^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\text{Q15aii } I_n = \int_0^1 x^n \sqrt{1-x^2} dx = \int_0^1 x^{n-1} x \sqrt{1-x^2} dx$$

Let $u = x^{n-1}$ and $\frac{dv}{dx} = x\sqrt{1-x^2}$,

$$\text{then } \frac{du}{dx} = (n-1)x^{n-2}, v = -\frac{(\sqrt{1-x^2})^3}{3}$$

$$I_n = [uv]_0^1 - \int_0^1 v \frac{du}{dx} dx = \left[-\frac{x^{n-1}(\sqrt{1-x^2})^3}{3} \right]_0^1 - \int_0^1 -\frac{(n-1)x^{n-2}(\sqrt{1-x^2})^3}{3} dx$$

$$= \frac{n-1}{3} \int_0^1 x^{n-2} (\sqrt{1-x^2})^3 dx = \frac{n-1}{3} \int_0^1 x^{n-2} (1-x^2) \sqrt{1-x^2} dx$$

$$= \frac{n-1}{3} \left(\int_0^1 x^{n-2} \sqrt{1-x^2} dx - \int_0^1 x^n \sqrt{1-x^2} dx \right) = \frac{n-1}{3} (I_{n-2} - I_n)$$

$$I_n = \frac{n-1}{3} (I_{n-2} - I_n), \therefore I_n = \left(\frac{n-1}{n+2} \right) I_{n-2}$$

$$\text{Q15aiii } I_5 = \left(\frac{5-1}{5+2} \right) I_3 = \left(\frac{4}{7} \right) \left(\frac{3-1}{3+2} \right) I_1 = \left(\frac{4}{7} \right) \left(\frac{2}{5} \right) \left(\frac{1}{3} \right) = \frac{8}{105}$$

$$\text{Q15bi } \frac{d}{dx} (\sqrt{x} + \sqrt{y}) = \frac{d}{dx} \sqrt{a}, \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}} = -\sqrt{\frac{d}{c}} \text{ at } P(c, d)$$

Equation of the tangent at $P(c, d)$:

$$y-d = -\sqrt{\frac{d}{c}}(x-c), \therefore y\sqrt{c} + x\sqrt{d} = d\sqrt{c} + c\sqrt{d}$$

Q15bii Let $y=0$, $x=\sqrt{cd}+c$; let $x=0$, $y=\sqrt{cd}+d$

$$OA+OB = x+y = c+2\sqrt{cd}+d = (\sqrt{c}+\sqrt{d})^2 = (\sqrt{a})^2 = a$$

Q15ci $P(x_1, y_1)$ is on both conics.

$$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x_1^2}{c^2} - \frac{y_1^2}{d^2} = 1$$

$$\therefore \frac{y_1^2}{d^2} + \frac{y_1^2}{b^2} = \frac{x_1^2}{c^2} - \frac{x_1^2}{a^2}$$

$$\therefore \left(\frac{1}{d^2} + \frac{1}{b^2} \right) y_1^2 = \left(\frac{1}{c^2} - \frac{1}{a^2} \right) x_1^2$$

$$\therefore \frac{x_1^2}{y_1^2} = \frac{a^2 c^2}{(a^2 - c^2)} \times \frac{(b^2 + d^2)}{b^2 d^2}$$



Q15cii Ellipse: $b^2 = a^2(1 - e^2)$, $\therefore ae = \sqrt{a^2 - b^2}$

$$m_e = \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} = -\frac{b^2 x_1}{a^2 y_1} \text{ at } P(x_1, y_1)$$

Hyperbola: $d^2 = c^2(E^2 - 1)$, $\therefore cE = \sqrt{c^2 + d^2}$

$$m_h = \frac{dy}{dx} = \frac{d^2 x}{c^2 y} = \frac{d^2 x_1}{c^2 y_1} \text{ at } P(x_1, y_1)$$

Both conics have the same foci, $\therefore ae = cE$, $\sqrt{a^2 - b^2} = \sqrt{c^2 + d^2}$

$$\therefore a^2 - b^2 = c^2 + d^2, \quad a^2 - c^2 = b^2 + d^2$$

$$\therefore \frac{x_1^2}{y_1^2} = \frac{a^2 c^2}{(a^2 - c^2)} \times \frac{(b^2 + d^2)}{b^2 d^2} = \frac{a^2 c^2}{b^2 d^2}$$

$$\therefore \text{the product of the two gradients} = m_e m_h = -\frac{b^2 d^2 x_1^2}{a^2 c^2 y_1^2} = -1$$

\therefore the tangents to the two conics at $P(x_1, y_1)$ are perpendicular.

Q16ai $\alpha = \cos \theta + i \sin \theta$

$$\begin{aligned} \alpha^k + \alpha^{-k} &= \cos k\theta + i \sin k\theta + \cos(-k\theta) + i \sin(-k\theta) \\ &= \cos k\theta + i \sin k\theta + \cos k\theta - i \sin k\theta \\ &= 2 \cos k\theta \end{aligned}$$

Q16aaii Geometric series: $a = \alpha^n$, $r = \alpha^{-1}$, number of terms = $2n + 1$

$$\begin{aligned} C &= \frac{\alpha^n(1 - (\alpha^{-1})^{2n+1})}{1 - \alpha^{-1}} = \frac{\alpha^n(1 - \alpha^{-2n-1})}{1 - \alpha^{-1}} \\ &= \frac{(1 - \alpha)(\alpha^n - \alpha^{-n-1})}{(1 - \alpha)(1 - \alpha^{-1})} = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1 - \alpha)(1 - \alpha^{-1})} \end{aligned}$$

$$\text{Q16aiii } \alpha^{-n} + \dots + \alpha^{-1} + 1 + \alpha + \alpha^n = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1 - \alpha)(1 - \alpha^{-1})}$$

$$1 + (\alpha + \alpha^{-1}) + (\alpha^2 + \alpha^{-2}) + \dots + (\alpha^n + \alpha^{-n}) = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{2 - (\alpha + \alpha^{-1})}$$

$$1 + 2 \cos \theta + 2 \cos 2\theta + \dots + 2 \cos n\theta = \frac{2 \cos n\theta - 2 \cos(n+1)\theta}{2 - 2 \cos \theta}$$

$$\therefore 1 + 2(\cos \theta + \cos 2\theta + \dots + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos \theta}$$

Q16aiv Let $\theta = \frac{\pi}{n}$.

$$\therefore 1 + 2 \left(\cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \dots + \cos \frac{n\pi}{n} \right) = \frac{\cos \frac{n\pi}{n} - \cos \frac{(n+1)\pi}{n}}{1 - \cos \frac{\pi}{n}}$$

$$\therefore 1 + 2 \left(\cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \dots + \cos \frac{n\pi}{n} \right) = \frac{-1 - \cos \left(\pi + \frac{\pi}{n} \right)}{1 - \cos \frac{\pi}{n}}$$

$$\therefore 1 + 2 \left(\cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \dots + \cos \frac{n\pi}{n} \right) = \frac{-1 + \cos \frac{\pi}{n}}{1 - \cos \frac{\pi}{n}} = -1$$

$$\therefore \cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \dots + \cos \frac{n\pi}{n} \text{ is independent of } n.$$

$$\text{Q16bi } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0$$

Vertices: $x = \pm a$; foci: $x = \pm 2a$

$$2a - a = 1 \text{ OR } 2a + a = 1$$

$$\therefore a = 1 \text{ OR } a = \frac{1}{3}$$

Q16ci

$$\begin{aligned} N(\text{colours for the second col}) &= N(\text{C same as B}) + N(\text{C diff. from B}) \\ &= N_s(\text{C}) \times N_s(\text{D}) + N_d(\text{C}) \times N_d(\text{D}) \\ &= 1(x-1) + (x-1)(x-1) \\ &= x^2 - 3x + 3 \end{aligned}$$

Q16cii For 1 column (2 by 1), it is given $N = x(x-1)$

$$N = x(x-1)(x^2 - 3x + 3)^{n-1} = x(x-1) \text{ for } n = 1$$

\therefore true for $n = 1$

Assuming $N = x(x-1)(x^2 - 3x + 3)^{n-1}$ is true for n columns.

For the next column, i.e. the $(n+1)$ th column, the number of ways is $x^2 - 3x + 3$.

\therefore the total number of ways to paint the 2 by $n+1$ grid

$$= x(x-1)(x^2 - 3x + 3)^{n-1} \times (x^2 - 3x + 3)$$

$$= x(x-1)(x^2 - 3x + 3)^n$$

\therefore true for $n \geq 1$

Q16ciii The total number of ways to paint the 2 by 5 grid using all 3 colours at least once

$$= N(n=5, x=3) - {}^3C_2 \times N(n=5, x=2)$$

$$= 3(3-1)(3^2 - 3 \times 3 + 3)^{5-1} - 3 \times 2(2-1)(2^2 - 3 \times 2 + 3)^{5-1}$$

$$= 480$$

Please inform mathline@itute.com re conceptual and/or mathematical errors.