Mathematics lesson – Binomial expansion algorithm (without calculator)
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The binomial theorem

This theorem is used to expand expression of the form $(p+q)^n$, where *n* is a positive integer.

$$(p+q)^n = {}^nC_0 p^n q^0 + {}^nC_1 p^{n-1} q^1 + {}^nC_2 p^{n-2} q^2 + \ldots\ldots + {}^nC_n p^0 q^n$$

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, …… can be determined by

$${}^nC_r = \frac{{}^nP_r}{r!}.$$

For example, ${}^7C_3 = \frac{{}^7P_3}{3!} = \frac{7\times6\times5}{3\times2\times1} = 35$.

Alternatively, for small *n*, the Pascal triangle can be used to evaluate those coefficients without using a calculator.

	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	1 7 21 35 35 21 7 1

For example, in the expansion of $(p+q)^7$, ${}^7C_0 = 1$, ${}^7C_1 = 7$, ${}^7C_2 = 21$, ${}^7C_3 = 35$, ……

Hence

$$(p+q)^7 = {}^7C_0p^7q^0 + {}^7C_1p^6q^1 + {}^7C_2p^5q^2 + {}^7C_3p^4q^3 + {}^7C_4p^3q^4 + {}^7C_5p^2q^5 + {}^7C_6p^1q^6 + {}^7C_7p^0q^7$$
$$= 1p^7q^0 + 7p^6q^1 + 21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 21p^2q^5 + 7p^1q^6 + 1p^0q^7$$

For large *n*, both manual methods become impractical and time consuming to evaluate the coefficients.

However, ccording to
$${}^nC_{r+1} = \frac{{}^nP_{r+1}}{(r+1)!} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)}{r!(r+1)} = \frac{{}^nC_r(n-r)}{r+1},$$
successive coefficients can be generated in a shorter time by

$${}^nC_{r+1} = \frac{{}^nC_r(n-r)}{r+1}.$$

For example, from ${}^7C_0 = 1$, ${}^7C_1 = \frac{{}^7C_0(7-0)}{0+1} = 7$,

${}^7C_2 = \frac{{}^7C_1(7-1)}{1+1} = \frac{7\times6}{2} = 21$, ${}^7C_3 = \frac{{}^7C_2(7-2)}{2+1} = \frac{21\times5}{3} = 35$,

${}^7C_4 = \frac{{}^7C_3(7-3)}{3+1} = \frac{35\times4}{4} = 35$, ${}^7C_5 = \frac{{}^7C_4(7-4)}{4+1} = \frac{35\times3}{5} = 21$,

${}^7C_6 = \frac{{}^7C_5(7-5)}{5+1} = \frac{21\times2}{6} = 7$, ${}^7C_7 = \frac{{}^7C_6(7-6)}{6+1} = \frac{7\times1}{7} = 1$.

To write down the terms in the expansion of $(p+q)^7$, follow the algorithm as shown below.

From the first term $1p^7q^0$,

the second term is $\frac{1\times7}{1}p^6q^1 = 7p^6q^1$,

the third term is $\frac{7\times6}{2}p^5q^2 = 21p^5q^2$,

the fourth term is $\frac{21\times5}{3}p^4q^3 = 35p^4q^3$ etc.

Using the same algorithm, the terms in the expansion of $(p+q)^{20}$ are:

$1p^{20}q^0$,

$\frac{1\times20}{1}p^{19}q^1 = 20p^{19}q^1$,

$\frac{20\times19}{2}p^{18}q^2 = 190p^{18}q^2$,

$\frac{190\times18}{3}p^{17}q^3 = 1140p^{17}q^3$,

$\frac{1140\times17}{4}p^{16}q^4 = 4845p^{16}q^4$,

$\frac{4845\times16}{5}p^{15}q^5 = 15504p^{15}q^5$ etc.

Instead of evaluating, e.g. ${}^{20}C_6$ by manual arithmetic:
${}^{20}C_6 = \frac{20\times19\times18\times17\times16\times15}{6\times5\times4\times3\times2\times1} = 38760$, evaluation time will be shortened by using the algorithm: $\frac{15504\times15}{6} = 38760$.