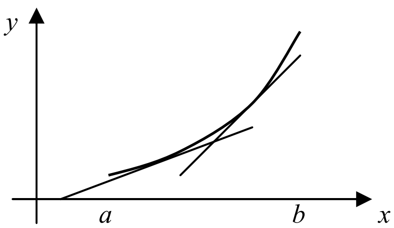
Inflection points

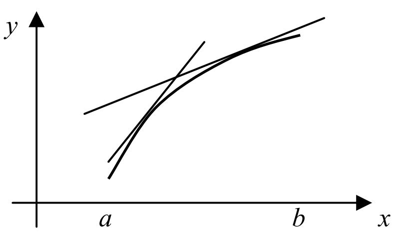
(Suitable for students in year 11/12)

Concave upward and concave downward

A curve is called concave upward on an interval [a,b] if it lies above the tangent drawn at any point in [a,b].

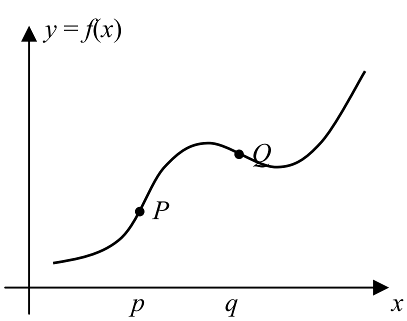


A curve is called concave downward on an interval [a,b] if it lies below the tangent drawn at any point in [a,b].



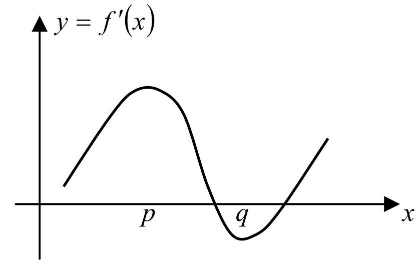
Inflection points

A point on a curve $y = f(x)$ is called an inflection point if the curve changes from concave upward to concave downward or from concave downward to concave upward at that point.



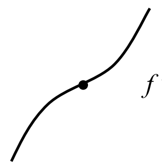
Points P (at $x = p$) and Q (at $x = q$) are inflection points of the curve. At P the curve changes from concave upward to concave downward as you trace the curve in the positive x-direction; at Q the curve changes from concave downward to concave upward.

The graph of the first derivative (gradient function) $y = f'(x)$ is shown below.

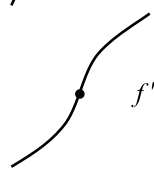


$y = f'(x)$ has turning points at $x = p$ and $x = q$,
$\therefore$ the values of the second derivatives at $x = p$ and $x = q$ are $f''(p) = 0$ and $f''(q) = 0$.

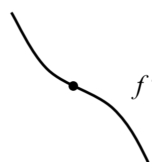
The following is a summary of possible inflection points.
Let $x = c$ be the x-coordinate of the inflection point. c^- means less than (or to the left of) c, c^+ means greater than (or to the right of) c.



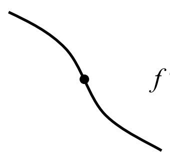
$f'(c^-) > 0$, $f'(c) > 0$, $f'(c^+) > 0$, $f''(c) = 0$.



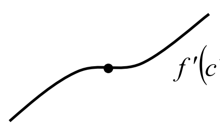
$f'(c^-) > 0$, $f'(c) > 0$, $f'(c^+) > 0$, $f''(c) = 0$.



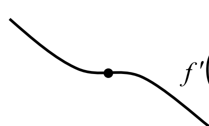
$f'(c^-) < 0$, $f'(c) < 0$, $f'(c^+) < 0$, $f''(c) = 0$.



$f'(c^-) < 0$, $f'(c) < 0$, $f'(c^+) < 0$, $f''(c) = 0$.



$f'(c^-) > 0$, $f'(c) = 0$, $f'(c^+) > 0$, $f''(c) = 0$.



$f'(c^-) < 0$, $f'(c) = 0$, $f'(c^+) < 0$, $f''(c) = 0$.

The common features of the possible inflection points are:
(1) $f''(c) = 0$, i.e. the second derivative of curve f at an inflection point equals zero.
(2) either both $f'(c^-)$ and $f'(c^+)$ are positive or both are negative.

In the last two examples above, $f'(c) = 0$, i.e. the gradient of the curve at the inflection point is zero. These inflection points are called **stationary inflection points**.

Steps in finding the inflection points of a function

Step 1. Find the plausible *x*-coordinate(s) of the inflection point(s) of *f* by letting $f''(x)=0$.

Step 2. Show that at a plausible *x*-coordinate, either both $f'(c^-)$ and $f'(c^+)$ are positive or both are negative for an inflection point at $x=c$.

Note: Step 2 is necessary because there are points where $f''(x)=0$ but they are not inflection points. See example 1.

If a further step is taken to show $f'(x)=0$, then the inflection point is stationary. See example 2.

Example 1 Find the *x*-coordinate(s) of the inflection point(s) of $f(x)=10x^7-14x^6+21x^5-35x^4$.

Step 1. $f(x)=10x^7-14x^6+21x^5-35x^4$,
$f'(x)=70x^6-84x^5+105x^4-140x^3$,
$f''(x)=420x^5-420x^4+420x^3-420x^2$.

Let $f''(x)=0$, i.e. $420x^5-420x^4+420x^3-420x^2=0$,
$420x^2(x^3-x^2+x-1)=0$,
$420x^2((x^3-x^2)+(x-1))=0$,
$420x^2(x^2(x-1)+1(x-1))=0$,
$420x^2(x-1)(x^2+1)=0$.
$\therefore x=0$ or $x=1$ are the plausible locations of inflection points.

Step 2. Check the gradients of the function to the immediate left and right of each point.

x	0^-	0^+	1^-	1^+
$f'(x)$	>0	<0	<0	<0

Since $f'(0^-)>0$ and $f'(0^+)<0$, $\therefore$ the point at $x=0$ is **not** an inflection point.

Since $f'(1^-)<0$ and $f'(1^+)<0$, both are negative, $\therefore$ the point at $x=1$ is an inflection point.

In general, for a function *f*, if the point at $x=c$ is an inflection point, then $f''(c)=0$.
However, the converse is not true. If $f''(c)=0$, the point at $x=c$ is **not** necessarily an inflection point.

Example 2 Find the *x*-coordinate(s) of the stationary inflection point(s) of $f(x)=8x^4+12x^3+6x^2+x$.

Step 1. $f(x)=8x^4+12x^3+6x^2+x$,
$f'(x)=32x^3+36x^2+12x+1$,
$f''(x)=96x^2+72x+12$.

Let $f''(x)=0$, i.e. $96x^2+72x+12=0$,
$12(8x^2+6x+1)=0$,
$12(4x+1)(2x+1)=0$.
$\therefore x=-\dfrac{1}{2}$ or $x=-\dfrac{1}{4}$ are the plausible locations of inflection points.

Step 2. Check the gradients of the function to the immediate left and right of each point, and at each point.

x	$\left(-\frac{1}{2}\right)^-$	$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^+$	$\left(-\frac{1}{4}\right)^-$	$-\frac{1}{4}$	$\left(-\frac{1}{4}\right)^+$
$f'(x)$	<0	0	<0	<0	<0	<0

Since both $f'\left(\left(-\frac{1}{4}\right)^-\right)$ and $f'\left(\left(-\frac{1}{4}\right)^+\right)$ are negative, $\therefore$ the point at $x=-\dfrac{1}{4}$ is an inflection point. However, it is not stationary because $f'\left(-\frac{1}{4}\right)\neq 0$.

Since both $f'\left(\left(-\frac{1}{2}\right)^-\right)$ and $f'\left(\left(-\frac{1}{2}\right)^+\right)$ are negative, and $f'\left(-\frac{1}{2}\right)=0$, $\therefore$ the point at $x=-\dfrac{1}{2}$ is a stationary inflection point.