Mathematics lesson – Finding Pythagorean Triads
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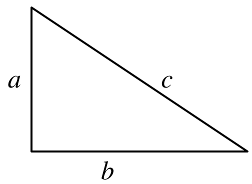
A Pythagorean triad is a set of 3 *natural numbers*, a, b and c, satisfying Pythagoras' theorem, $a^2 + b^2 = c^2$ for a right-angled triangle.



3 is the lowest natural number in a Pythagorean triad.
In fact, there is only one triad, namely $\{3,4,5\}$ containing a 3 in it.
Hence, the three natural numbers $a, b, c \geq 3$.

How do you find a triad (or triads) containing a particular natural number (≥ 3)?

Consider the following discussion:

$(p-q)^2 + 4pq = (p+q)^2$,

$\therefore \left(\frac{p-q}{2}\right)^2 + pq = \left(\frac{p+q}{2}\right)^2$, where p and q are natural numbers and $p > q$.

For $\frac{p+q}{2}$ and $\frac{p-q}{2}$ to be natural numbers, either both p and q are odd or both are even numbers.

Let $a = \frac{p-q}{2}$ and $c = \frac{p+q}{2}$.

Then a, b and c form a triad if $b^2 = pq$, i.e. $b = \sqrt{pq}$.

From the above discussion, the quickest way to find a triad (or triads) containing a particular natural number as one of the two smaller numbers is to let b be the particular number, then find all possible values of p and q, such that $pq = b^2$, $p > q$ and either both are odd or both are even.

If the particular number is the largest member c of the triad, then find all possible values of p and q, such that $\frac{p+q}{2} = c$, $p > q$, either both are odd or both are even, and pq is a perfect square.

Example 1 Find all distinct Pythagorean triads containing the number 5.

If 5 is one of the two smaller numbers, then $pq = b^2 = 25$. The only possible values for p and q are $p = 25$ and $q = 1$.

$\therefore a = \frac{p-q}{2} = \frac{25-1}{2} = 12$, $c = \frac{p+q}{2} = 13$. $\{12, 5, 13\}$.

If 5 is the largest member c, then $\frac{p+q}{2} = 5$, $p + q = 10$.

Possible p and q values:

p	q	$a = \frac{p-q}{2}$	$b = \sqrt{pq}$	Triad
9	1	4	3	$\{4, 3, 5\}$
8	2	3	4	$\{3, 4, 5\}$
7	3	2	$\sqrt{21}$	-
6	4	1	$\sqrt{24}$	-

Hence there are only two distinct Pythagorean triads, namely $\{3, 4, 5\}$ and $\{12, 5, 13\}$.

Example 2 Find all distinct Pythagorean triads containing the number 56.

If 56 is one of the two smaller numbers, then $pq = b^2 = 3136$.
Possible values:

p	q	$a = \frac{p-q}{2}$	$c = \frac{p+q}{2}$	Triad
1568	2	783	785	$\{783, 56, 785\}$
784	4	390	394	$\{390, 56, 394\}$
392	8	192	200	$\{192, 56, 200\}$
224	14	105	119	$\{105, 56, 119\}$
196	16	90	106	$\{90, 56, 106\}$
112	28	42	70	$\{42, 56, 70\}$
98	32	33	65	$\{33, 56, 65\}$

Note: Only those yielding a triad are shown in the above table.

If 56 is the largest member c, then $\frac{p+q}{2} = 56$, $p + q = 112$.

Possible p and q values:

p	q	$a = \frac{p-q}{2}$	$b = \sqrt{pq}$	Triad
111	1	55	$\sqrt{111}$	-
110	2	54	$\sqrt{220}$	-
109	3	53	$\sqrt{327}$	-
108	4	52	$\sqrt{432}$	-
:	:	:	:	:
:	:	:	:	:
57	55	1	$\sqrt{3135}$	-

Use calculator to find that none of the products pq in the table above is a perfect square. There is no triad with 56 as the largest member.

Hence there are only seven triads as shown in the first table.