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Mathematical Methods

2008

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

Consider the function $f: D \to R$, $f(x) = \tan\left(\frac{x}{2}\right)$ and $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which one of the following statements is true?

A. f has asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

B. f has asymptotes at $x = -\pi$ and $x = \pi$.

C. f has no asymptotes.

D. f has asymptotes at $x = -2\pi$ and $x = 2\pi$.

E. f has asymptotes at $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

Question 2

Given $b = 2\log_2\left(\frac{a}{2}\right)$, $a =$

A. $e^{\frac{1}{2}(b+2)\log_e 2}$ B. $2e^{\frac{1}{2}(b+2)}$ C. $2^{\frac{1}{2}(b+1)}$ D. $2^{\frac{1}{2}(b-1)}$ E. $e^{\frac{b}{2}\log_e 2+1}$

Question 3

The equation $e^{2x} - 2e^x + k = 0$ has two solutions when

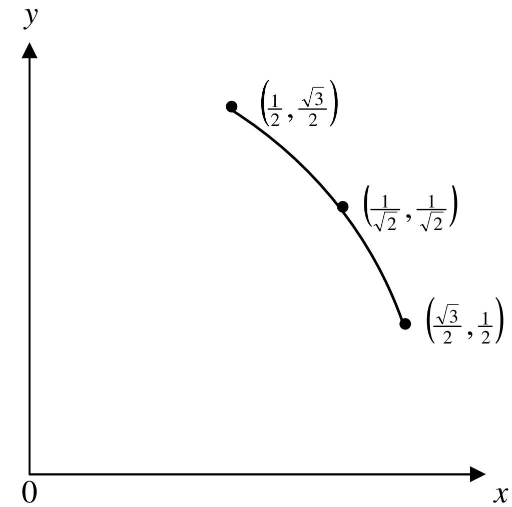
A. $k \in R$ B. $k \in R^+$ C. $k \in R^-$ D. $k \in (-\infty, 1)$ E. $k \in (0.3, 0.7)$

Question 4

If $f(x) = 2\cos(3x)$, then $\frac{1}{4} f\left(\frac{\pi}{6} - \frac{1}{3}x\right) =$

A. $\frac{1}{2}\cos(x)$ B. $\frac{1}{2}\sin(x)$ C. $\frac{1}{2}\cos\left(\frac{\pi}{6} - \frac{1}{3}x\right)$ D. $\frac{1}{2}\sin\left(\frac{\pi}{6} - \frac{1}{3}x\right)$ E. $\frac{1}{2}\sin\left(\frac{1}{3}x\right)$

Question 5



The function shown in the graph is in the form $f(x) = a\sqrt{1-bx^2}$. The values of a and b are respectively

A. 1 and 4 B. 4 and 1 C. 1 and 2 D. 1 and 1 E. 2 and $\frac{1}{4}$

Question 6

For $a, b, c \in R^+$, the range of the function $\left|a - be^{-x}\right| - c$ is

A. $[a-c, \infty)$ B. $[0, \infty)$ C. $(-\infty, b+c]$ D. $[-c, \infty)$ E. $(-\infty, c]$

Question 7

Which one of the following expressions **cannot** be expressed as a single polynomial function of x over its entire domain?

A. $\left(3\sqrt{x} + x\right)\left(3\sqrt{x} - x\right)$

B. $\left(1 - x\sqrt{2}\right)\left(2 + x\sqrt{2}\right)$

C. $\sqrt[3]{x^3 - 3x^2 + 3x - 1}$

D. $\dfrac{x^{\frac{3}{2}} - (2x)^{\frac{5}{2}}}{x^{-\frac{3}{2}}}$

E. $\left|x^4 - x^2 - 1\right|$

Question 8

Which one of the following **cannot** be the possible number of intersections of a function and its inverse?

A. 0 B. 1 C. 2 D. No more than 2 E. Any number

Question 9

If $f: R \to R,\ f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g: [1,\infty) \to R,\ g(x) = \frac{1}{x} - 1$, the range of $f \circ g$ is

A. $[0,1)$ B. $(0,1]$ C. $[-2,0]$ D. $[0,-2]$ E. $[0,-2)$

Question 10

Given $f(x) = 1 - \sqrt{x}$, which one of the following statements is true?

A. $f(xy) + f\left(\frac{x}{y}\right) = 2f(x)$

B. $f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right) = 2[f(x) - f(y)]$

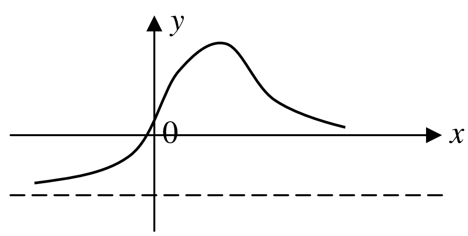
C. $f(xy) + f\left(\frac{x}{y}\right) = f(x)f(y)$

D. $f\left(\frac{x}{y}\right) = \frac{f(x) - f(y)}{1 - f(y)}$

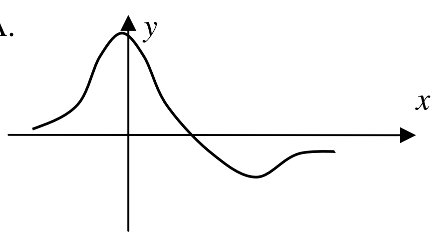
E. $f(2x) - f(2y) = [f(x) - f(y)].[f(x) + f(y)]$

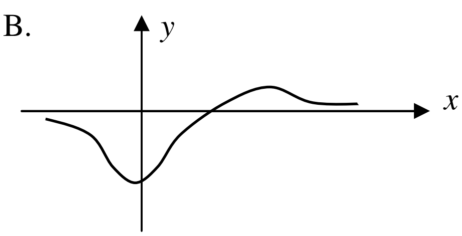
Question 11

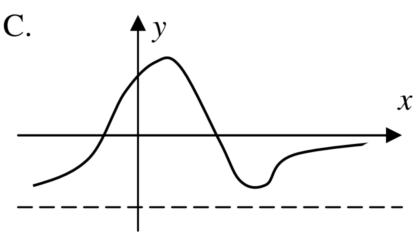
For graph of $y = f(x)$ is shown below.

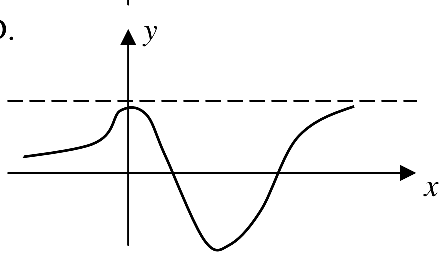


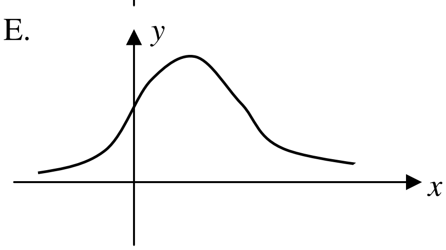
Which one of the following is the graph of $y = f'(x)$?

A.


B.


C.


D.


E.


Question 12

Given $P(x)=\dfrac{f(\log_e x)}{g(\sqrt{x})}$, $P'(x)=$

A. $\dfrac{g(\sqrt{x})f'(\log_e x)-f(\log_e x)g'(\sqrt{x})}{2x\sqrt{x}g(x)}$

B. $\dfrac{2\sqrt{x}g(\sqrt{x})f'(\log_e x)-xf(\log_e x)g'(\sqrt{x})}{2x\sqrt{x}\left[g(\sqrt{x})\right]^2}$

C. $\dfrac{g(\sqrt{x})f'(\log_e x)-f(\log_e x)g'(\sqrt{x})}{2x\sqrt{x}\left[g(\sqrt{x})\right]^2}$

D. $\dfrac{g(x)f'(x)-f(x)g'(x)}{2x\sqrt{x}[g(x)]^2}$

E. $\dfrac{2\sqrt{x}g(x)f'(x)-xf(x)g'(x)}{2x\sqrt{x}[g(x)]^2}$

Question 13

The equation of the normal to the curve $y=2x^3-3ax^2+5$ at $x=a$ is

A. $y=x$ B. $y=0$ C. $x=0$ D. $y=a$ E. $x=a$

Question 14

The gradient of the tangent to the curve $y=\dfrac{x\tan\left(\sqrt{xe^x}\right)}{2\pi}$ at $x=\dfrac{3}{4}$ is closest to

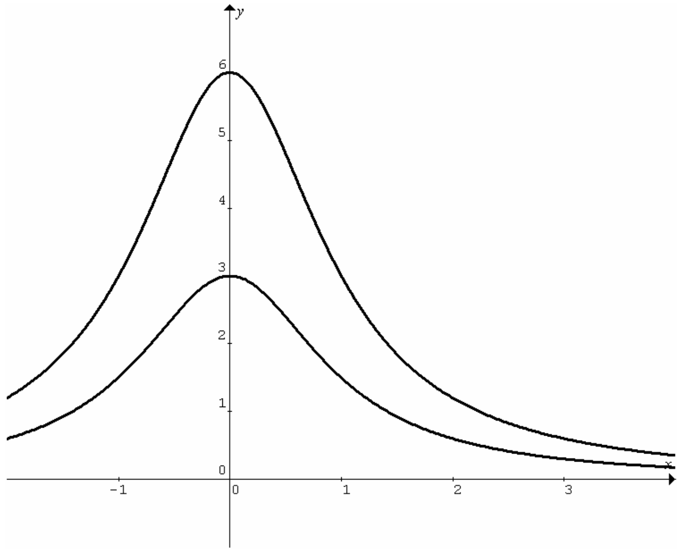
A. 2.373 B. 2.372 C. 23.41 D. 23.42 E. 3.668

Question 15

Using the approximation $f(x+h)\approx f(x)+hf'(x)$ for $f(x)=\sqrt{1-x}$, the approximate value of $f(-3.1)$ is

A. $\dfrac{79}{40}$ B. $\dfrac{81}{40}$ C. $\dfrac{197}{100}$ D. $\dfrac{2531}{1250}$ E. $\dfrac{4937}{2500}$

Question 16



Using 'right rectangles' of unit width, the approximate area bounded by $x=-1$, $x=3$ and the two curves $y=f(x)$ and $y=\frac{1}{2}f(x)$ shown in the figure above is closest to

A. 5.4 B. 6.0 C. 6.6 D. 10.8 E. 13.2

Question 17

Given the continuous and **decreasing** function $f(x)=(x+2)g(x)$, and $F(x)=\int f(x)dx$, the area bounded by the *x*-axis, $x=a$, $x=b$ and $y=f(x)$ for $a<-2<b$ is

A. $2F(-2)-F(a)-F(b)$

B. $F(b)-F(a)$

C. $F(a)-F(b)$

D. $F(b)+F(a)$

E. $2F(-2)+F(a)-F(b)$

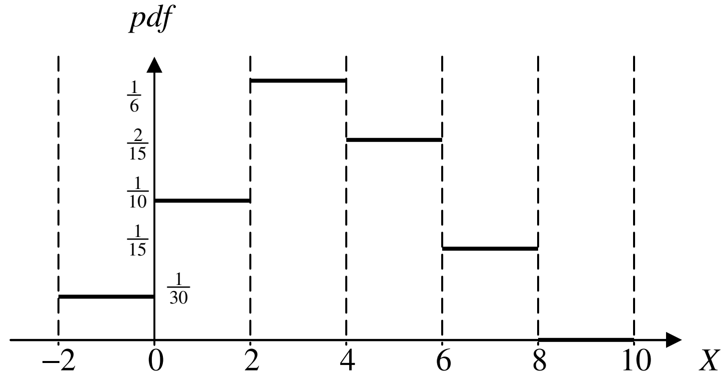
Question 18

Roll a fair die twice.

The probability of getting the difference of the two uppermost numbers greater than 1 is

A. $\frac{2}{3}$ B. $\frac{5}{6}$ C. $\frac{5}{9}$ D. $\frac{11}{18}$ E. $\frac{13}{18}$

Question 19



The graph above shows a probability distribution of random variable *X*. The value of $\overline{X}$ equals

A. $\frac{51}{15}$ B. $\frac{53}{15}$ C. $\frac{51}{30}$ D. $\frac{53}{30}$ E. 3

Question 20

	$X > 0.36$	$X < 0.36$	
$X > 0.64$	0.19		
$X < 0.64$		0.48	
			1

The above is a probability table for continuous random variable *X* with some missing probability values. When $X < 0.64$, the probability that $X > 0.36$ equals

A. $\frac{11}{27}$ B. $\frac{13}{25}$ C. $\frac{33}{100}$ D. $\frac{33}{52}$ E. $\frac{19}{48}$

Question 21

The heights of the children in a queue for an amusement park ride are normally distributed. 95.4% of the children in the queue have heights between 1.25 m and 1.35 m, i.e. within $\mu \pm 2\sigma$. The percentage of the children in the queue with heights between 1.28 m and 1.38 m is closest to

A. 95% B. 90% C. 85% D. 80% E. 75%

Question 22

The waiting time *t* (hours) to see a doctor has a probability distribution given by the probability density function $f(t) = \frac{1}{\sqrt{3}\cos^2 t}$, $0 \le t \le \frac{\pi}{3}$. The median waiting time *t* is closest to

A. 0.9 B. 0.8 C. 0.7 D. 0.6 E. 0.5

SECTION 2 Extended-answer questions

Instructions for Section 2

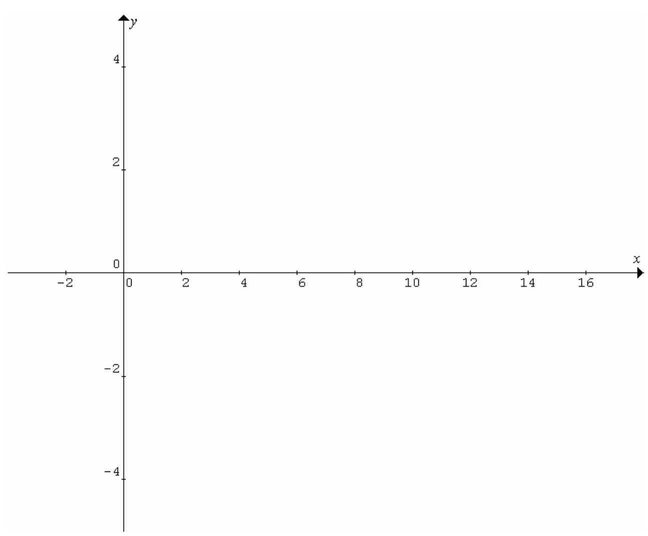
Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Question 1

Consider the function $f(x)=\frac{16}{x^4}-1$. Length is measured in metres and time in seconds.

a. Show that the equations of the inverse of $f(x)$ is $y=\pm\frac{2}{(x+1)^{\frac{1}{4}}}$. 2 marks

b. Sketch the graph of the inverse of $f(x)$. Label the asymptote(s) and intercept(s).



2 marks

c. Let L be the length of a vertical line touching the inverse of $f(x)$. Express L in terms of x.

1 mark

The vertical line is at $x = 0$ when $t = 0$, and it moves in the positive x direction at a speed of 2 metres per second.

d. i. Show that the rate of decrease of L is $\dfrac{2}{(x+1)^{\frac{5}{4}}}$.

d. ii. Find the rate of decrease of L when $t = 7.5$.

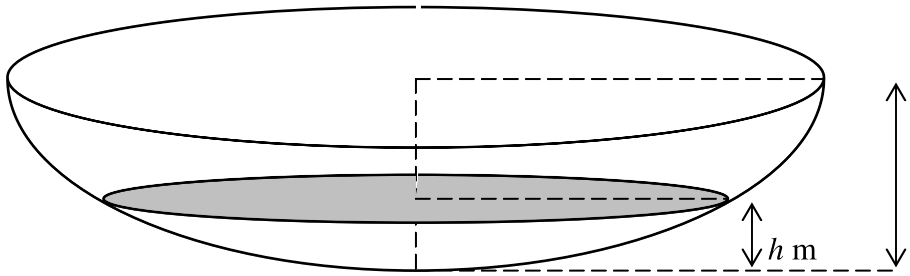
2 + 1 = 3 marks

e. Find the exact average rate of increase of the area swept by the vertical line in the time interval $[0,7.5]$.

3 marks

Total 11 marks

Question 2



The diagram above shows a pond with water filled to a depth of h m. The shaded part is the water surface and has an area $A = 2\pi\left[1-(1-h)^2\right]$ m^2. The volume of water is $V = 2\pi\left[\frac{2}{3}-(1-h)+\frac{(1-h)^3}{3}\right]$ m^3.

a. Factorise $2\pi\left[1-(1-h)^2\right]$ completely.

1 mark

b. Expand $\frac{2}{3}-(1-h)+\frac{(1-h)^3}{3}$.

1 mark

c. Find the maximum value of V.

1 mark

The pond is initially full. Water is drained from the pond at 2 litres per second. (Note: 1 litre = 1000 cm^3)

d. How long (in seconds) will it take to empty the pond?

1 mark

e. Find the rate of decrease of the depth h with respect to t when $h = 0.5$ m.

3 marks

f. Hence find the rate of decrease of the area A with respect to t when $h = 0.5$ m.

2 marks

The draining is stopped when $h = 0.5$ m, and pebbles of volume 0.831 m^3 are placed (submerged) at the bottom of the pond.

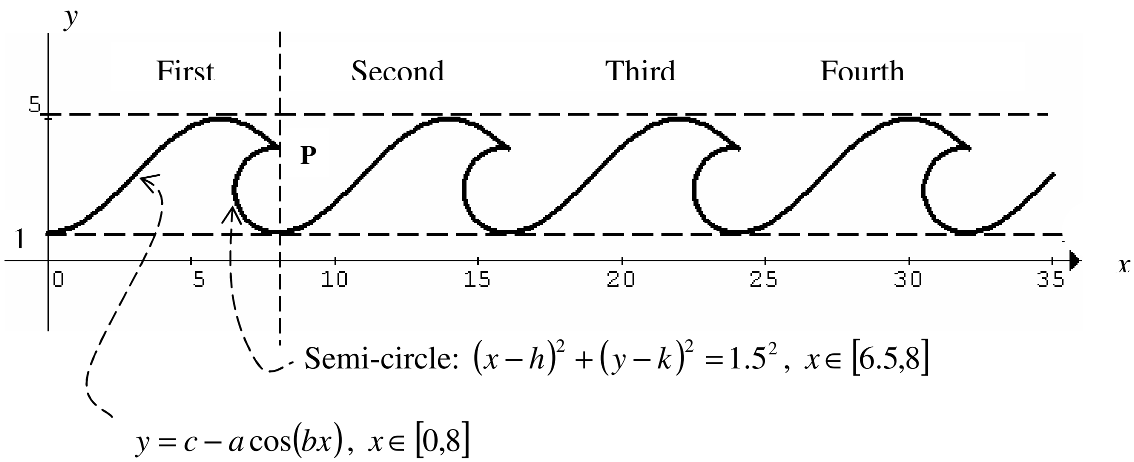
g. i. Determine the height (3 decimal places in m) of the water surface from the bottom of the pond.

ii. Does the rate of decrease of the depth h with respect to t change when draining is resumed? Explain.

2 + 2 = 4 marks

Total 13 marks

Question 3
The following drawing shows the cross-section of four successive identical wave crests. **The seabed is at** $y = -2$. Length is measured in metres.



a. Find the exact values of a, b, c, h and k.

3 marks

b. Write down the exact coordinates of point P, the intersection of the cosine curve and the semi-circle.

1 mark

c. i. Find the equation (including its domain) of the semi-circle in the third wave crest.

ii. Find the equation (including its domain) of the cosine curve in the fourth wave crest.

2 + 2 = 4 marks

d. i. Find the exact cross-sectional area of the second wave crest above the seabed.

d. ii. Find the volume of water (nearest m^3) above the seabed contained in the second wave crest when the wave front is 10 m long, assuming the wave crest has a uniform cross-section along the wave front.

3 + 1 = 4 marks

Total 12 marks

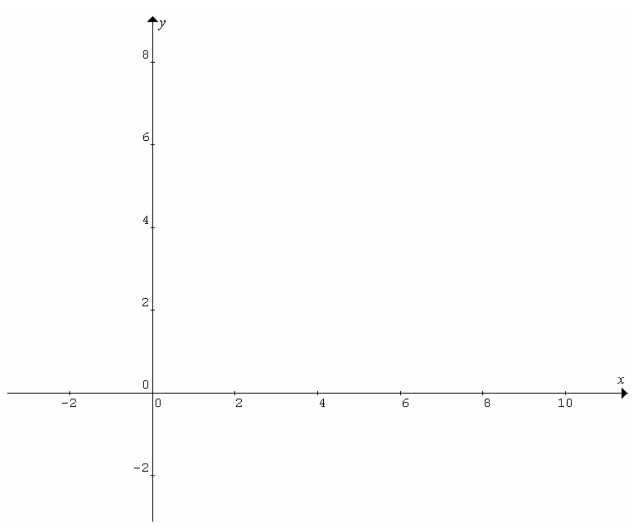
Question 4

Consider $f(x) = a\log_e(x) + \frac{1}{2}$.

a. Find $f^{-1}(x)$.

2 marks

b. Let $a = 2$. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.



2 marks

c. i. Find the value of a (3 decimal places) such that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ touch each other once only.

c. ii. Find the coordinates (3 decimal places) of the contact point.

3 + 1 = 4 marks

Total 8 marks

Question 5

A machine is calibrated to fill cans with **375.5 ml** of soft drink. The volume of soft drink (x ml) in a can is a random variable X with a probability distribution given by probability density function $f(x) = e^{-\pi(x-375.5)^2}$.

a. Out of 1000 filled cans of soft drink, how many are expected to be over 375 ml?

2 marks

692 out of 1000 cans are over **375.3** ml without changing the original setting of the machine. The manager of the soft drink company changes the setting of the machine to fill cans with ***k* ml** so that 692 out of 1000 cans are over **375** ml.

b. Find the value of k, assuming the standard deviation remains the same after the change.

3 marks

The company has 5 such machines at the new setting. The manager takes a can at random from each machine.

c. What is the probability (3 decimal places) that 4 of them are under 375 ml?

2 marks

The manager makes another inspection and takes **two** cans at random from each machine.

d. What is the probability (3 decimal places) that 8 of the 10 cans are under 375 ml?

1 mark

e. The manager measured 3 of them and found 2 cans under 375 ml. What is the probability (3 decimal places) that 8 of the 10 cans are under 375 ml?

2 marks

Another company has only two machines, A and B. The volume of soft drink in a can filled by machine A is a random variable with a normal distribution, $\mu = 375.5$ ml and $\sigma = 0.3989$ ml. The volume of soft drink filled by machine B is a random variable with a normal distribution of the same mean $\mu = 375.5$ ml but different standard deviation $\sigma = 0.7978$ ml. The two machines produce **the same number** of cans of soft drink. All cans of soft drink produced by the two machines are now put together. Let the volume of drink in a can be random variable Y. **Information**: For a normal distribution,
$\Pr(\mu - \sigma < X < \mu + \sigma) = 0.683$, $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = 0.954$ and $\Pr(\mu - 3\sigma < X < \mu + 3\sigma) = 0.997$.

f. i. Describe the shape of the probability distribution of Y.

f. ii. It is true that $\Pr(374.15 < Y < 376.85) = 0.954$. Is Y normally distributed? Justify your answer.

1 + 3 = 4 marks

Total 14 marks

End of exam 2