2008 NSW BOS Mathematics Extension 1 Solutions
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Q1a Let $p(x)=x^3$. When $p(x)$ is divided by $x+3$, the remainder is $p(-3)=(-3)^3=-27$.

Q1b $\frac{d}{dx}\cos^{-1}(3x)=\frac{-1}{\sqrt{1-(3x)^2}}=\frac{-3}{\sqrt{1-9x^2}}$

Q1c $\int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}dx=\left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1}=\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{-1}{2}=\frac{\pi}{3}$

Q1d $(2x+3y)^{12}=........+{}^{12}C_4(2x)^8(3y)^4+........$
The coefficient is ${}^{12}C_4(2^8)(3^4)=10264320$.

Q1e Let $u=\sin\theta$, $\frac{du}{d\theta}=\cos\theta$.

$\int_0^{\frac{\pi}{4}}\cos\theta\sin^2\theta d\theta=\int_0^{\frac{\pi}{4}}u^2\frac{du}{d\theta}d\theta=\int_0^{\frac{1}{\sqrt{2}}}u^2du=\left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$

$=\frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3=\frac{\sqrt{2}}{12}$.

Q1f $(x-3)(5-x)>0$ when $3<x<5$. The domain of $f(x)=\log_e[(x-3)(5-x)]$ is the interval $(3,5)$.

Q2a Let $u=\log_e x$, $\frac{du}{dx}=\frac{1}{x}$.

$\int_e^{e^2}\frac{1}{x(\log_e x)^2}dx=\int_e^{e^2}\frac{1}{u^2}\frac{du}{dx}dx=\int_1^2\frac{1}{u^2}du=\left[-\frac{1}{u}\right]_1^2=-\frac{1}{2}+1=\frac{1}{2}$

Q2b Given $\ddot{x}=x+4$, and $v=0$ at $x=1$.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x+4$, $\frac{1}{2}v^2=\int(x+4)dx=\frac{(x+4)^2}{2}+c$.

$0=\frac{25}{2}+c$, $c=-\frac{25}{2}$.

$\therefore\frac{1}{2}v^2=\frac{(x+4)^2}{2}-\frac{25}{2}$, $v^2=(x+4)^2-25$.

At $x=2$, $v^2=11$, speed $=|v|=\sqrt{11}$.

Q2c $p(x)=ax^3+16x^2+cx-120$,
$p(-2)=a(-2)^3+16(-2)^2+c(-2)-120=0$, $\therefore 4a+c=-28$ ..(1)
$p(3)=a(3)^3+16(3)^2+c(3)-120=0$, $\therefore 9a+c=-8$ ………….(2)
(2) – (1): $5a=20$, $a=4$ and $c=-44$.
$\therefore p(x)=4x^3+16x^2-44x-120=4(x^3+4x^2-11x-30)$
$(-2)(3)(\alpha)=30$, $\therefore\alpha=-5$.

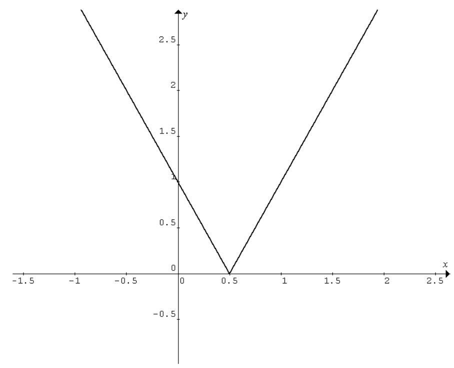
Q2d $f(x)=\tan x-\log_e x$ has a zero near $x=4$.

$f'(x)=\sec^2 x-\frac{1}{x}$.

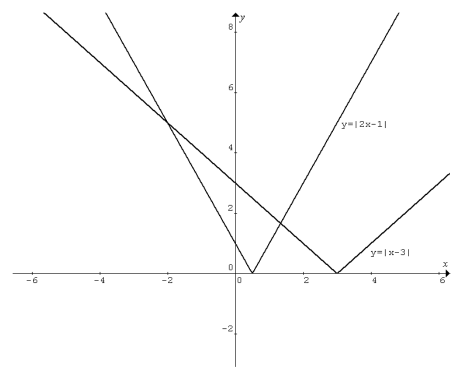
Newton's method: $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

Another approximation $=4-\frac{f(4)}{f'(4)}=4-\frac{-0.2285}{2.0906}=4.11$

Q3ai



Q3aii



$-(x-3)=-(2x-1)$, $-x+3=-2x+1$, $x=-2$

$-(x-3)=2x-1$, $-x+3=2x-1$, $3x=4$, $x=\frac{4}{3}$

$\therefore|2x-1|\le|x-3|$ when $-2\le x\le\frac{4}{3}$.

Q3b When $n=1$, $\frac{1}{6}(1+1)(2(1)+7)=3=1\times3$, the statement is true. Assume that it is true when $n=k$,

i.e. $1\times3+2\times4+3\times5+...+k(k+2)=\frac{k}{6}(k+1)(2k+7)$, then

$1\times3+...+k(k+2)+(k+1)(k+3)=\frac{k}{6}(k+1)(2k+7)+(k+1)(k+3)$

$=(k+1)\left(\frac{2k^2+13k+18}{6}\right)=\frac{(k+1)}{6}(k+2)(2k+9)$

$=\frac{(k+1)}{6}((k+1)+1)(2(k+1)+7)$. $\therefore$ it is also true for $n=k+1$.

Hence it is true for all $n\ge1$.

Q3ci $\theta = \tan^{-1}\left(\frac{x}{\ell}\right)$, $\frac{d\theta}{dt} = \frac{\ell}{\ell^2 + x^2} \times \frac{dx}{dt} = \frac{v\ell}{\ell^2 + x^2}$, where $v = \frac{dx}{dt}$.

Q3cii At $x = 0$, $m = \frac{v}{\ell}$ is the maximum value of $\frac{d\theta}{dt}$.

Q3ciii $\frac{d\theta}{dt} = \frac{m}{4}$, $\frac{v\ell}{\ell^2 + x^2} = \frac{v}{4\ell}$, $\therefore \ell^2 + x^2 = 4\ell^2$, $\frac{x}{\ell} = \pm\sqrt{3}$, $\therefore \theta = \tan^{-1}\left(\pm\sqrt{3}\right) = \pm\frac{\pi}{3}$.

Q4ai $T = 190 - 185e^{-kt}$. At $t = 0$, $T = 190 - 185e^{0} = 5$.
$\frac{dT}{dt} = 185ke^{-kt} = -k\left(-185e^{-kt}\right) = -k(T - 190)$.

Q4aii At $t = 0$ (9 am), $T = 5$.
At $t = 1$ (10 am), $T = 29$.
$\therefore 29 = 190 - 185e^{-k}$, $185e^{-k} = 161$, $e^{-k} = \frac{161}{185}$.

When $T = 80$, $80 = 190 - 185\left(\frac{161}{185}\right)^{t}$,
$\left(\frac{161}{185}\right)^{t} = \frac{110}{185}$, $t = \frac{\log_e \frac{110}{185}}{\log_e \frac{161}{185}} = 3.74142 \text{ h} \approx 3 \text{ h } 44 \text{ min}$
The turkey is cooked at 12:44 pm.

Q4bi Group Barbara and John (in that order) together and arrange with the other six people.
Number of ways $= 7! = 5040$.

Q4bii Total number of ways without restrictions $= 8!$
Half of these John goes through after Barbara, and the other half Barbara goes through after John.
Number of ways $= \frac{8!}{2} = 20160$.

Q4ci Gradient of $QO = \frac{aq^2}{2aq} = \frac{q}{2}$.

Gradient of $PT = \left.\frac{dy}{dx}\right|_{x=2ap} = p$.

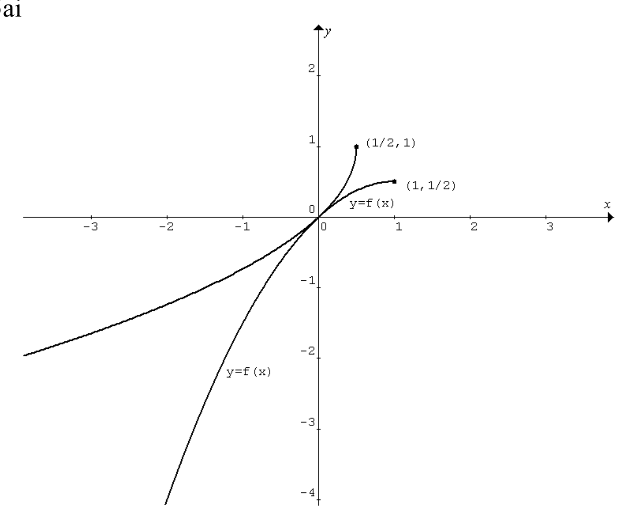
$PT \perp QO$, $\therefore p \times \frac{q}{2} = -1$, $pq = -2$.

Q4cii Gradient of $PL = \frac{p}{2}$.

Gradient of $QT = q$. $\frac{p}{2} \times q = \frac{pq}{2} = \frac{-2}{2} = -1$,
$\therefore PL$ and QT are perpendicular, $\angle PLQ = 90°$.

Q4ciii Since $\angle PLQ = \angle PKQ = 90°$,
$\therefore PQ$ is a diameter of a circle through L and K.
M is the midpoint of PQ,
$\therefore MK = ML$, radius of the circle.

Q5ai

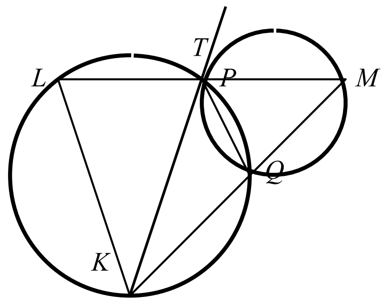


Q5aii The inverse of $y = x - \frac{1}{2}x^2$ is $x = y - \frac{1}{2}y^2$.
$y^2 - 2y + 2x = 0$, $y^2 - 2y + 1 - (1 - 2x) = 0$, $(y - 1)^2 = 1 - 2x$,
$y - 1 = -\sqrt{1 - 2x}$.
$\therefore y = 1 - \sqrt{1 - 2x}$, $\therefore f^{-1}(x) = 1 - \sqrt{1 - 2x}$.
Note: Minus is chosen because the domain of $f(x)$ is the range of $f^{-1}(x)$.

Q5aiii $f^{-1}\left(\frac{3}{8}\right) = 1 - \sqrt{1 - 2 \times \frac{3}{8}} = 1 - \frac{1}{2} = \frac{1}{2}$.

Q5b SHM: $\ddot{x} = -k^2 x$, where k is a positive constant.
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -k^2 x$, $\frac{1}{2}v^2 = -\frac{k^2x^2}{2} + c$.
Maximum speed = 2 ms^{-1} when $x = 0$, $\therefore c = 2$, $v^2 = 4 - k^2x^2$.
When $v = 0$, $0 = 4 - k^2x^2$, $x = \pm\frac{2}{k}$. Amplitude $= \frac{2}{k}$.
At $x = -\frac{2}{k}$, maximum acceleration $6 = -k^2\left(-\frac{2}{k}\right)$, $\therefore k = 3$.
Hence the amplitude $= \frac{2}{k} = \frac{2}{3}$ metres,
And the period $= \frac{2\pi}{k} = \frac{2\pi}{3}$ seconds.

Q5c



$\angle PMQ = \angle KPQ$ (Angle between tangent and chord equals angle subtended by chord at a point on circumference)

$\angle PMQ + \angle MPQ = \angle KPQ + \angle MPQ$

$\angle PQK = \angle KPM$ ($\angle PQK$ is an exterior angle of ΔPMQ)

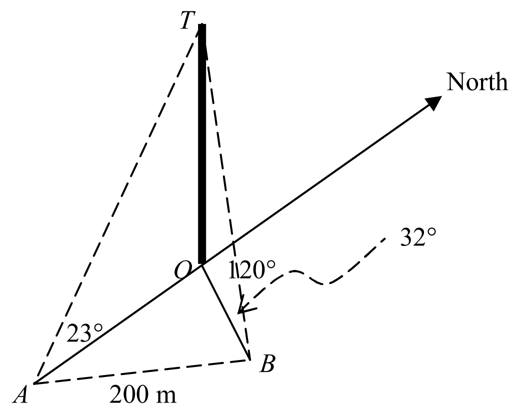
$\angle PQK + \angle KLP = \angle KPM + \angle KPL = 180°$

($\angle PQK + \angle KLP$ is the sum of the opposite angles of a quadrilateral with vertices on the circle)

$\therefore \angle KLP = \angle KPL$

$\therefore \Delta PKL$ is isosceles.

Q6ai



Q6aii Let h m be the height of the tower.

$OA = h\tan 23°$, $OB = h\tan 32°$, $\angle AOB = 180° - 120° = 60°$.

The cosine rule:

$$200^2 = \left(\frac{h}{\tan 23°}\right)^2 + \left(\frac{h}{\tan 32°}\right)^2 - 2\left(\frac{h}{\tan 23°}\right)\left(\frac{h}{\tan 32°}\right)\cos 60°$$

$$40000 = \frac{h^2}{0.18018} + \frac{h^2}{0.39046} - \frac{h^2}{0.26524}, \ h \approx 96 \text{ m.}$$

Q6b $\sin 3\theta + \sin 2\theta = \sin\theta$,

$\therefore \sin 3\theta = -\sin 2\theta + \sin\theta = -2\sin\theta\cos\theta + \sin\theta$

$= -\sin\theta(2\cos\theta - 1)$.

Given $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 4\sin\theta - 4\sin^3\theta - \sin\theta$

$= \sin\theta(4\cos^2\theta - 1)$.

$\therefore \sin\theta(4\cos^2\theta - 1) = -\sin\theta(2\cos\theta - 1)$,

$\therefore \sin\theta(4\cos^2\theta - 1) + \sin\theta(2\cos\theta - 1) = 0$,

$2\sin\theta(2\cos^2\theta + \cos\theta - 1) = 0$.

Hence $\sin\theta = 0$, $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$, where $0 \le \theta \le 2\pi$.

$\therefore \theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ or 2π.

Q6ci

$(1+x)^{p+q} = 1 + {}^{p+q}C_1 x + {}^{p+q}C_2 x^2 + \ldots + {}^{p+q}C_q x^q + \ldots + {}^{p+q}C_{p+q-1} x^{p+q-1} + x^{p+q}$

The term of $\frac{(1+x)^{p+q}}{x^q}$ which is independent of x is ${}^{p+q}C_q$.

Q6cii $(1+x)^p = 1 + {}^pC_1 x + {}^pC_2 x^2 + \ldots + {}^pC_i x^i + \ldots + {}^pC_{p-1} x^{p-1} + x^p$

$\left(1 + \frac{1}{x}\right)^q = 1 + {}^qC_1 \frac{1}{x} + {}^qC_2 \frac{1}{x^2} + \ldots + {}^qC_i \frac{1}{x^i} + \ldots + {}^qC_{q-1} \frac{1}{x^{q-1}} + \frac{1}{x^q}$,

where $p \le q$.

In the expansion of $(1+x)^p\left(1 + \frac{1}{x}\right)^q$, the term independent of x is

$1^2 + {}^pC_1 {}^qC_1 x \frac{1}{x} + {}^pC_2 {}^qC_2 x^2 \frac{1}{x^2} + \ldots + {}^pC_p {}^qC_p x^p \frac{1}{x^p}$,

i.e. $1^2 + {}^pC_1 {}^qC_1 + {}^pC_2 {}^qC_2 + \ldots + {}^pC_p {}^qC_p$.

$\therefore 1^2 + {}^pC_1 {}^qC_1 + {}^pC_2 {}^qC_2 + \ldots + {}^pC_p {}^qC_p = {}^{p+q}C_q$.

Q7a $y = Vt\sin\theta - \frac{1}{2}gt^2$. When $y = h$, $\frac{1}{2}gt^2 - Vt\sin\theta + h = 0$,

$$\therefore t_1 = \frac{V\sin\theta - \sqrt{(V\sin\theta)^2 - 2gh}}{g}$$

and $t_2 = \frac{V\sin\theta + \sqrt{(V\sin\theta)^2 - 2gh}}{g}$.

$$\therefore t_1 + t_2 = \frac{2V}{g}\sin\theta \text{ and } t_1 t_2 = \frac{(V\sin\theta)^2 - (V\sin\theta)^2 + 2gh}{g^2} = \frac{2h}{g}.$$

Q7b $\tan\alpha + \tan\beta = \frac{h}{Vt_1\cos\theta} + \frac{h}{Vt_2\cos\theta} = \frac{h}{V\cos\theta}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$

$$= \frac{h}{V\cos\theta}\left(\frac{t_1 + t_2}{t_1 t_2}\right) = \frac{h}{V\cos\theta}\left(\frac{\frac{2V}{g}\sin\theta}{\frac{2h}{g}}\right) = \tan\theta.$$

Q7c $\tan\alpha\tan\beta = \left(\frac{h}{Vt_1\cos\theta}\right)\left(\frac{h}{Vt_2\cos\theta}\right) = \frac{h^2}{V^2 t_1 t_2 \cos^2\theta}$

$$= \frac{h^2}{V^2\left(\frac{2h}{g}\right)\cos^2\theta} = \frac{gh}{2V^2\cos^2\theta}.$$

Q7d From the given diagram:

$r = OP + PN = h\cot\alpha + h\cot\beta = h(\cot\alpha + \cot\beta)$ and

$w = PN - QN = PN - OP = h\cot\beta - h\cot\alpha = h(\cot\beta - \cot\alpha)$.

Q7e $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$.

Eliminate t to obtain $y = x\tan\theta - \frac{gx^2}{2(V\cos\theta)^2}$.

$\frac{dy}{dx} = \tan\theta - \frac{gx}{(V\cos\theta)^2}$.

At L, $t = t_1 = \frac{V\sin\theta - \sqrt{(V\sin\theta)^2 - 2gh}}{g}$,

$\therefore \frac{x}{V\cos\theta} = \frac{V\sin\theta - \sqrt{(V\sin\theta)^2 - 2gh}}{g}$,

$\therefore \frac{gx}{(V\cos\theta)^2} = \frac{V\sin\theta - \sqrt{(V\sin\theta)^2 - 2gh}}{V\cos\theta} = \tan\theta - \frac{\sqrt{(V\sin\theta)^2 - 2gh}}{V\cos\theta}$

$\therefore \frac{dy}{dx} = \frac{\sqrt{(V\sin\theta)^2 - 2gh}}{V\cos\theta}$,

i.e. $\tan\phi = \frac{\sqrt{(V\sin\theta)^2 - 2gh}}{V\cos\theta}$.

$\tan\alpha - \tan\beta = \frac{h}{Vt_1\cos\theta} - \frac{h}{Vt_2\cos\theta} = \frac{h}{V\cos\theta}\left(\frac{1}{t_1} - \frac{1}{t_2}\right)$

$= \frac{h}{V\cos\theta}\left(\frac{t_1 - t_2}{t_1 t_2}\right) = \frac{h}{V\cos\theta}\left(\frac{\frac{2}{g}\sqrt{(V\sin\theta)^2 - 2gh}}{\frac{2h}{g}}\right)$

$= \frac{\sqrt{(V\sin\theta)^2 - 2gh}}{V\cos\theta}$.

$\therefore \tan\phi = \tan\alpha - \tan\beta$.

Q7f $\frac{\tan\theta}{\tan\phi} = \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \frac{\frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta}}{\frac{\tan\alpha - \tan\beta}{\tan\alpha\tan\beta}} = \frac{\frac{1}{\tan\beta} + \frac{1}{\tan\alpha}}{\frac{1}{\tan\beta} - \frac{1}{\tan\alpha}}$

$= \frac{\cot\beta + \cot\alpha}{\cot\beta - \cot\alpha} = \frac{\frac{r}{h}}{\frac{w}{h}} = \frac{r}{w}$.

$\therefore \frac{w}{\tan\phi} = \frac{r}{\tan\theta}$.

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