Physics notes – Einstein's special relativity

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Maxwell's Prediction

A changing electric field produces a changing magnetic field and a changing magnetic field produces a changing electric field. Maxwell found that the net result of these interacting changing fields was a wave of electric and magnetic fields that propagate through space. Waves produced in this manner are called **electromagnetic waves**. They satisfy a set of equations called **Maxwell's equations**. They travel through space with a speed of 3.00×10^8 ms^{-1}. This value was calculated from $v=\frac{1}{\sqrt{\mu_o\varepsilon_o}}$, a natural outcome of Maxwell's equations, where

$\mu_o=4\pi\times10^{-7}$ TmA^{-1} is the **permeability of free space**,

$\varepsilon_o=8.85\times10^{-12}$ C^{2}N^{-1}m^{-2} is the **permittivity of free space**.

This calculated value is precisely equal to the measured speed of light. Maxwell argued that light must be an electromagnetic wave and have a speed in vacuum given by $v=\frac{1}{\sqrt{\mu_o\varepsilon_o}}$. If light travels in a material medium with permeability μ and permittivity ε, then its speed becomes $v=\frac{1}{\sqrt{\mu\varepsilon}}$.

Since $v=\frac{1}{\sqrt{\mu_o\varepsilon_o}}$ does not refer to any specific frame of reference, Maxwell predicted that the speed of light does not depend on the speed of the source or the speed of the medium.

At that time scientists believed that waves required a carrier (medium) to propagate. Maxwell was puzzled by the question of what might be the carrier of electromagnetic waves in regions free of any known matter. Maxwell called the mysterious carrier the **ether**. It was assumed to permeate all space and be absolutely at rest. It was presumed that the speed of light predicted by Maxwell must be relative to this ether. Light seemed to single out one frame of reference (the ether), which was better than any other.

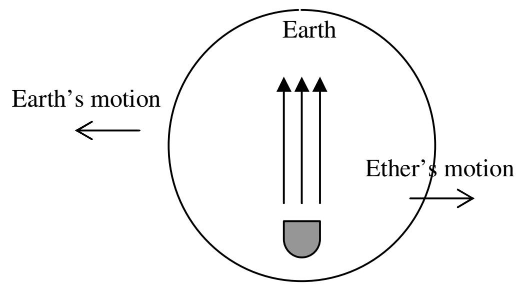
It appeared that Maxwell's predictions contradicted the relativity principles of Galileo and Newton:

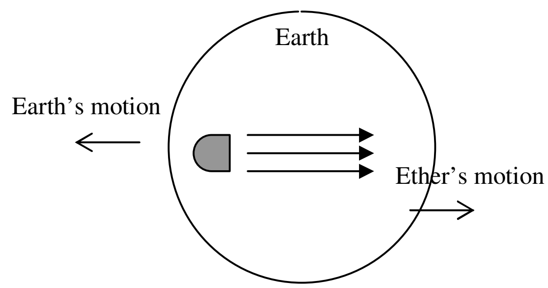
(1) There is no **absolute frame of reference**, i.e. all **inertial frames of reference** are equivalent for the description of mechanical phenomena. No one inertial frame of reference is any better than another.
(2) All velocity measurements are relative to the frame of reference.

In late 19th century, A. A. Michelson and E. W. Morley designed an experiment to find this absolute frame of reference, the ether, one that could be considered to be at rest.

The Michelson-Morley experiment

The earth revolves around the sun at a speed of approximately 3×10^4 ms^{-1} through the ether that was **assumed** to be in existence. Hence the ether appears to move in the opposite direction at the same speed. They sent two light beams through the ether, one parallel and one perpendicular to the motion of the earth.





They expected to find the speeds of the two light beams to be different. However, they found no significant difference in the two speeds. To explain this **null result** was a great challenge at that time.

The apparent inconsistencies between the electromagnetic theory and Newtonian mechanics, and the puzzling null result of the Michelson-Morley experiment were resolved by Einstein when he published his special theory of relativity in 1905.

The postulates of Einstein's special theory of relativity

(1) The laws of physics are the same in all inertial frames of reference.
(2) The speed of light has a constant value for all observers regardless of their motion or the motion of the source.

These two postulates form the foundation of Einstein's special theory of relativity.

Postulate (1) was an extension of the Newtonian model to include not only the laws of mechanics but also those of the rest of physics, including electricity and magnetism.

Postulate (2) was completely different from that of Newton. The notion of relative velocity was discarded and replaced with the idea that the speed of light in vacuum is always the same, no matter what the speed of the observer or the source. Hence the existence of the ether as the absolute frame of reference was no longer required.

Interpretation of the null result of the M-M experiment in terms of the postulates of Einstein's special relativity

The Michelson-Morley experiment was intended to determine the existence of an absolute frame of reference. Postulate (1) implies that there is no such preferred frame and thus explains the null result of the M-M experiment. The M-M experiment can be considered as evidence for postulate (1).

Postulate (2) says that the speed of light through empty space is always the same, independent of the state of motion of the observer. It is fully consistent with the finding in M-M experiment that there was no significant difference in the two speeds.

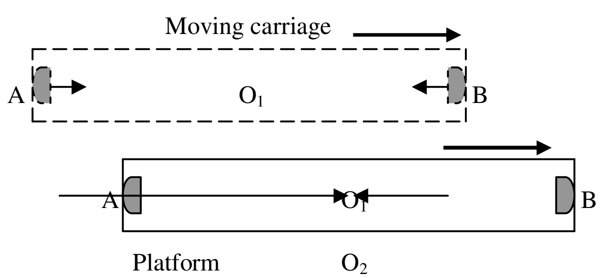
Einstein's theory requires us to give up commonsense intuitive ideas about space and time. It leads to some strange but interesting consequences. We will follow Einstein's technique in exploring these consequences, i.e. by means of **thought experiments**.

A thought experiment on simultaneity

Two events are simultaneous if they occur at exactly the same time. It is easy to tell if they occur at the same place. But it is harder to know if the two events occur at a great distance from each other because it takes time for the signal (light) to travel the distance. Two events that appear to occur simultaneously are not simultaneous, in fact the far away event occurs earlier.

One of the consequences of Einstein's theory is that time is no longer regarded as absolute. The question of whether two events are simultaneous or not depends on the observer's frame of reference.

Example 1 Imagine that there are two people A and B flashing their flashlights, one at each end of a long train carriage that passes a station platform at high speed. Suppose that there are two observers, observer O_1 is at the midpoint of A and B in the carriage and observer O_2 stands at the platform. Suppose O_1 receives the lights at the same time when O_1 and O_2 are directly opposite each other, therefore to observer O_1 the flashing of light by A and B are simultaneous.



To observer O_2, for both lights reaching O_1 (and O_2 because they are directly opposite each other) at the same time, A must flash the flashlight earlier to allow for the longer distance of travel to reach O_1 since O_1 is carried forward by the train.

Thus two events which are simultaneous to one observer are not necessarily simultaneous to another observer.

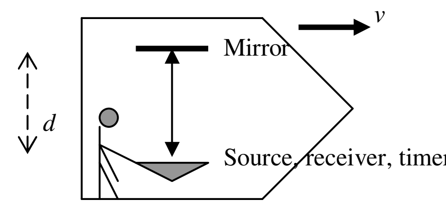
Example 2 Suppose only one of the lights reaches O_2 when O_1 and O_2 are directly opposite each other. Observer O_2 calculates back to find that A and B flash the flashlights simultaneously. Which light reaches O_2 first, and how do they appear to O_1?

Flashlight from B reaches O_2 first because O_2 is closer to B.

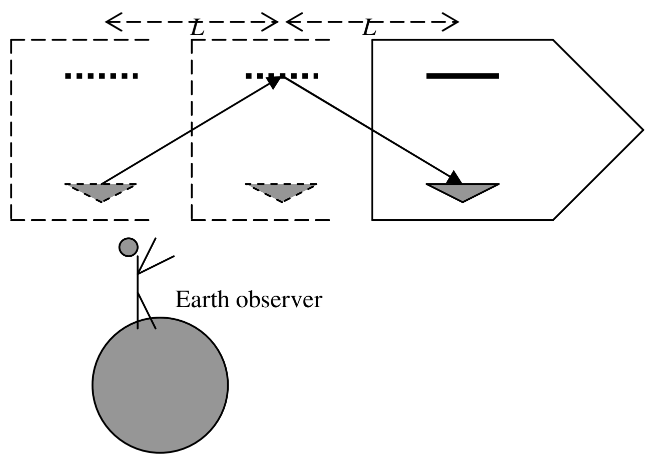
The light from B also reaches O_1 first because O_1 and O_2 are directly opposite each other. To observer O_1, B must flash the flashlight earlier because O_1 is equidistant from A and B. Hence the flashing of light by A and B are not simultaneous to O_1.

A thought experiment to show the elapse of time occurs at different rates depending on the motion of an observer relative to an event – Time dilation

Time dilation can be shown by a thought experiment: imagine a spaceship travelling past the earth at high speed v, the time it takes for light to travel forwards and return inside a spaceship is longer for an earth observer Δt than for an observer on the spaceship Δt_0.



Observer on spaceship



The result of this thought experiment is $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where c is the speed of light in vacuum 3.0×10^8 ms^{-1}, which is a universal constant according to Einstein's postulate (2) in his special theory of relativity. $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is known as Lorentz factor.

Since $\sqrt{1-\frac{v^2}{c^2}}$ is always less than 1, $\therefore \Delta t > \Delta t_0$, i.e. the time between the two events (the sending of the light, the reception of the light) is greater for the earth observer than for the observer travelling with the source and receiver. This is known as time dilation, a consequence of Einstein's postulates. Time is actually measured to pass more slowly in any moving frame of reference as compared to your own.

Example 1 Show that $\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, i.e. $\Delta t = \gamma \Delta t_0$.

Observer on spaceship: $\Delta t_0 = \frac{2d}{c}$.

Earth observer: $\Delta t = \frac{2\sqrt{d^2+L^2}}{c} = \frac{2\sqrt{d^2+\left(v\frac{\Delta t}{2}\right)^2}}{c}$,

$\Delta t = \sqrt{\frac{4d^2+(v\Delta t)^2}{c^2}}$, $(\Delta t)^2 = \frac{4d^2}{c^2} + \frac{v^2}{c^2}(\Delta t)^2$,

$(\Delta t)^2 - \frac{v^2}{c^2}(\Delta t)^2 = \frac{4d^2}{c^2}$, $\Delta t\sqrt{1-\frac{v^2}{c^2}} = \frac{2d}{c} = \Delta t_0$, $\therefore \Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

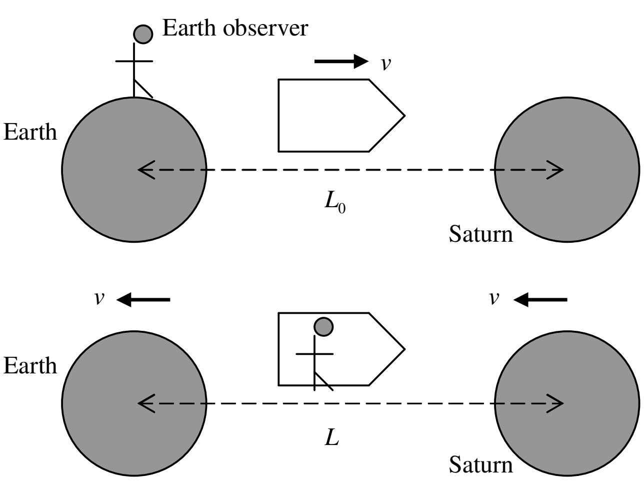
Example 2 The mean lifetime of an elementary particle called muon is 2.2×10^{-6} s as measured in a frame of reference that it is at rest. What will its mean lifetime be as measured in an earth laboratory if it is travelling at half of the speed of light relative to the earth?

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.2\times10^{-6}}{\sqrt{1-\frac{(0.5c)^2}{c^2}}} = \frac{2.2\times10^{-6}}{\sqrt{0.75}} = 2.5\times10^{-6}\text{ s.}$$

Time interval that is measured in the frame of reference in which objects are at rest is known as **proper time**. In example 2 above 2.2×10^{-6} s is the proper time, and in example 1 it is Δt_0.

A thought experiment to show that spatial measurements are different when measured in different frames of reference

The length of an object is measured to be shorter when it is moving than when it is at rest. This is known as **length contraction** and can be shown by a thought experiment: imagine an observer on earth watching a spaceship travelling at high speed v from earth to, say Saturn. The distance between earth and Saturn is L_0 (measured by the earth observer). The time required for the trip calculated by the earth observer is $\Delta t = \frac{L_0}{v}$.



For an observer on the spacecraft, the spacecraft is at rest but earth and Saturn move with speed v. The time required for the trip (i.e. the time interval between the two events, departure of earth and arrival of Saturn) is the proper time and should be less than Δt, time measured by the earth observer, i.e. $\Delta t_0 = \Delta t\sqrt{1-\frac{v^2}{c^2}}$. Hence the distance between the two planets calculated by the observer on the spaceship is $L = v\Delta t_0$.

$\therefore L = v\Delta t\sqrt{1-\frac{v^2}{c^2}}$, $\therefore L = L_0\sqrt{1-\frac{v^2}{c^2}}$. Hence $L < L_0$.

The last equation and inequality apply to lengths of objects as well as to distance between objects.

Length contraction occurs only along the direction of motion.

If an object (or two objects) is at rest relative to an observer, the length of the object (or distance between two objects) is known as **proper length**.

To the earth observer, earth and Saturn are at rest. The distance L_0 (measured by the earth observer) between earth and Saturn is the **proper length**.

Example 1 A spacecraft of proper length 50.0 m moves at 3.0×10^7 ms^{-1} relative to the earth. By what percent of its proper length will it appear to be shortened to an earth observer?

$$\left(\frac{L_0-L}{L_0}\right)\times100\% = \left(1-\frac{L}{L_0}\right)\times100\% = \left(1-\sqrt{1-\frac{v^2}{c^2}}\right)\times100\%$$

$$=\left(1-\sqrt{1-0.1^2}\right)\times100\% \approx 0.5\%$$

Example 2 Two cars A and B pass each other at a speed of 0.18 times the speed of light. A person in car B says her car is 6.00 m long and car A is 6.15 m long. What does a person in car A measure for these two lengths?

Proper length of car A: $6.15 = L_{0A}\sqrt{1-0.18^2}$, $L_{0A} = 6.25$ m.

Relativistic length of car B: $L_B = 6.00\times\sqrt{1-0.18^2} = 5.90$ m.

Example 3 A star is 27 light-years away according to earth observers. (Note: A light-year is the distance travelled by light in an earth year.) How long would it take a spaceship travelling at 0.90c to reach the star from earth as measured by observers (a) on earth; (b) on the spaceship? (c) What is the distance travelled according to observers on the spaceship? (d) What will the spaceship observers calculate their speed to be?

(a) $\Delta t = \frac{L_0}{v} = \frac{27c}{0.90c} = 30$ years.

(b) $\Delta t_0 = \Delta t\sqrt{1-\frac{v^2}{c^2}} = 30\times\sqrt{1-0.90^2} \approx 13$ years.

(c) $L = v\Delta t_0 = 0.90c\times13 \approx 12c$, i.e. 12 light years.

(d) $v = \frac{L}{\Delta t_0} = 0.90c$.

Relativistic mass

Time intervals and lengths are shown to be relativistic, i.e. their values depend on the frame of reference from which they are measured. Mass of an object is also a relativistic quantity. According to the special theory of relativity, the mass of an object increases as its speed increases, and the relativistic mass is $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0$.

In this mass-increases formula, m_0 represents the mass of the object measured in a frame of reference in which it is at rest, and it is called the **rest mass** of the object; *m* represents the mass of the object measured in a frame of reference in which it moves at speed *v*.

The ultimate speed

A consequence of Einstein's special theory of relativity is that objects cannot keep on increasing their speed to or past the speed of light, i.e. $v < c$.

As $v \to c$, $m \to \infty$. To accelerate an object to speed *c* would require an infinite amount of energy, and so is impossible.

If $v > c$, $\sqrt{1-\frac{v^2}{c^2}}$ is undefined as a real factor and so lengths, time intervals and mass for ordinary objects would not be real quantities.

Relativistic mass of an object and the energy equivalent

When a net force is applied to an object at rest (rest mass m_0), the object increases in speed and hence in mass. Work is done on the object during the application of the net force. It can be shown that the work done is $W = mc^2 - m_0c^2$, where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

According to the work-energy theorem, the work done must equal the kinetic energy of the object since it started from rest, and if no other forms of mechanical energy are involved.
$\therefore KE = mc^2 - m_0c^2$ or $(\gamma - 1)m_0c^2$.

This equation led Einstein to the idea that mass is a form of energy, i.e. mass and energy are equivalent.

The quantity $mass \times c^2$ is called the **mass energy** of an object. So mc^2 is the mass energy *E* of the object when it moves at speed *v*, and m_0c^2 is the mass energy of the object when it is at rest, $\therefore$ it is also called the **rest energy** of the object. Hence

$$E = mc^2 \\ = m_0c^2 + KE.$$

Work done on an object increases the mass energy of the object because it gains *KE*.

Note: At high speed ($v \to c$), $KE \neq \frac{1}{2}mv^2$, $KE \neq \frac{m_0v^2}{2\sqrt{1-\frac{v^2}{c^2}}}$.

Example 1 An elementary particle has a rest mass of 2.4×10^{-28} kg. It is accelerated to a speed of 0.75*c*. Calculate (a) its mass (b) its mass energy and (c) its kinetic energy at this speed.

(a) $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.4\times10^{-28}}{\sqrt{1-0.75^2}} = 3.6\times10^{-28}$ kg.

(b) $E = mc^2 = 3.3\times10^{-11}$ J.

(c) $KE = mc^2 - m_0c^2 = (m - m_0)c^2 = 1.1\times10^{-11}$ J.

Example 2 Calculate the relativistic mass of an object moving at 40,000 kmh^{-1} (escape velocity from earth), given its rest mass is 1.00 kg.

40000 kmh^{-1} $= \frac{40000}{3.6} = 11111.11$ ms^{-1}

$= \frac{11111.11}{3.0\times10^8} = 3.70\times10^{-5}c$.

$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1.00}{\sqrt{1-(3.70\times10^{-5})^2}} = 1.00$ kg (1.000000001 kg)

Example 3 In the Stanford Linear Accelerator electrons are accelerated to very high speed. In terms of *c* what is the speed that makes the relativistic mass of an electron 10,000 times its rest mass?

$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, $\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Let $v = xc$, $10000 = \frac{1}{\sqrt{1-x^2}}$, $1 - x^2 = 10^{-8}$, $x^2 = 1 - 10^{-8}$,
$x = 0.999999995$, $\therefore v = 0.999999995\,c$.

Mass and energy conversions

Mass and energy are equivalent. For this idea to have any practical meaning, then mass ought to be convertible to energy and vice versa, just as the different types of energy are inter-convertible. This has been experimentally confirmed in nuclear and elementary particle processes.

Energy released = Difference in total rest energy before and after process

Example 1 The Sun is powered by the nuclear reaction $4H \to He$, where four hydrogen nuclei fuse to form a helium nucleus. The rest mass of a hydrogen nucleus is 1.673×10^{-27} kg and the rest mass of a helium nucleus is 6.645×10^{-27} kg. How much radiation energy is released by this nuclear reaction? [Physics(Pilot)]

Energy released = Difference in total rest energy before and after process

$= [4(1.673\times10^{-27}) - 6.645\times10^{-27}](3.0\times10^8)^2$

$= 4.23\times10^{-12}$ J.

