Physics notes – Motion in 1 and 2 dimensions

Free download and print from www.itute.com

Frames of reference A frame of reference is a set of coordinate axes fixed to some body (or group of bodies) such as the earth, a moving train, the moon, spacecraft etc. Every measurement must be made with respect to a frame of reference. Many measurements are made with respect to the earth and it should be stated as the frame of reference. However, it is not always specified for the sake of simplicity.

Example 1 Consider a train travelling at 80 km h^{-1} in the same direction as a car travelling at 100 km h^{-1}. Both speeds have the same frame of reference, i.e. the earth (the ground). They are recorded by an observer standing on the ground. An observer in the moving car sees the train moving backwards at 20 km h^{-1}. In this case the frame of reference used to measure the speed of the train is the moving car. The earth is considered as a stationary frame of reference while the car is a moving frame of reference.

Can you describe the motion of the car seen by an observer in the train?

Everyday motion can be explained in terms of the **Newtonian model**.

The three laws in the Newtonian model

Newton's first law (also known as the *law of inertia*): If an object experiences no net force due to other bodies, it either remains at rest or remains in motion at the same speed in a straight line.

The tendency of an object to maintain its state of motion (i.e. at rest or in uniform straight-line motion) is called its *inertia*.

A frame of reference, in which Newton's first law is valid, is called an *inertial frame of reference*. Stationary and moving (at constant speed in a straight line, i.e. at constant velocity) frames of reference are inertial frames of reference.

A *noninertial frame of reference* is one in which Newton's first law does not hold. A frame of reference which is moving with increasing or decreasing speed and/or in a curve is a noninertial frame of reference.

Newton's second law: The acceleration of an object is directly proportional to the net force exerted on it and is inversely proportional to its mass. The direction of the acceleration is the same as the direction of the net force.

$$a \propto F_{net}$$

$$a \propto \frac{1}{m}$$

$\vec{a}$ and $\vec{F}_{net}$ are in the same direction,

i.e. $\vec{a} = \dfrac{\vec{F}_{net}}{m}$ or $\vec{F}_{net} = m\vec{a}$.

Example 1 An empty truck has a maximum acceleration of 3 ms^{-2}.
(a) What is its maximum acceleration when it is loaded with goods two times its mass?
(b) What is the value of the ratio, the mass of the loaded truck M to the mass of the empty truck m, if the maximum acceleration is 2ms^{-2} when loaded?

(a) The total mass is 3 times the mass when the truck is empty.
$\therefore$ the acceleration is $\frac{1}{3}$ of the acceleration when the truck is empty, i.e. $\frac{1}{3}\times 3 = 1\,\text{ms}^{-2}$.
(b) $M \times 2 = m \times 3$, $\therefore \dfrac{M}{m} = \dfrac{3}{2}$.

Friction

Without friction between the car tyres and the road surface, a car cannot change its velocity, i.e. no acceleration is possible.

During braking, the friction force on the tyres is opposite to the direction of motion. When the car is speeding up, the friction force is in the direction of motion. This is the force which propels the car forward and it is called the driving force. When the car is turning at constant speed, the friction force is perpendicular to the direction of motion (given by the velocity vector) and it is towards the centre of the turn.

Example 1 The wheels of a car of mass 1200 kg were locked in a sudden braking. It came to a stop from a speed of 30 ms^{-1} in 15 m. Determine the average friction force between each tyre and the road surface.

$v^2 = u^2 + 2a_{av}s$, $0 = 30^2 + 2a_{av}(15)$, $a_{av} = {}^{-}30$ ms^{-2}.
$\vec{F}_{net,av} = m\vec{a}_{av} = 1200 \times {}^{-}30 = {}^{-}36000$ N (Negative sign means the friction force is opposite to the direction of motion).

For each tyre, average friction force $= {}^{-}9000$ N.

Example 2 A 1.5 tonnes car towing a 1.5 tonnes caravan has an acceleration of 1.2 ms^{-2}. The total resistance to the motion of the car is 100 N, and to the motion of the caravan is 120 N.
(a) Calculate the driving force of the car.
(b) Calculate the tension in the towbar.

(a) Consider the car and the caravan together. The driving force of the car overcomes the resistance and causes both the car and the caravan to accelerate.
$\vec{F}_{net} = \vec{F}_{drive} + {}^{-}220$, total mass $m = 1500 + 1500 = 3000$ kg.
$\vec{F}_{net} = m\vec{a}$, $F_{drive} - 220 = 3000 \times 1.2$, $F_{drive} \approx 3.8 \times 10^3$ N.

(b) Consider the caravan alone. The force of the towbar overcomes the resistance on the caravan and causes it to accelerate.
$\vec{F}_{net} = \vec{F}_{towbar} + {}^{-}120$, $\vec{F}_{net} = m\vec{a}$, $\vec{F}_{towbar} + {}^{-}120 = 1500 \times {}^{+}1.2$,
$F_{towbar} \approx 1.9 \times 10^3$ N.

Example 3 A 2.0 tonnes truck slows down uniformly from 30 ms^{-1} to 20 ms^{-1} in 3.0 s.
(a) Calculate the restraining force of a fastened seatbelt on the 80 kg driver.
(b) What is the restraining force of a seatbelt on a 40 kg passenger?
(c) The passenger holds a helium balloon inside the cabin with the windows closed. In which direction will the balloon move during the slow down?

(a) $v = u + at$, $20 = 30 + a \times 3.0$, $a = \frac{^{-}10}{3.0}$.

Consider the 80 kg driver: $\vec{F}_{net} = m\vec{a}$,

$\vec{F}_{belt} = 80 \times \frac{^{-}10}{3.0} \approx {}^{-}2.7 \times 10^2$ N ($^{-}$267 N)

(b) Consider the 40 kg passenger: Since $F_{net} \propto m$ and the mass of the passenger is $\frac{1}{2}$ of the mass of the driver, $\therefore \vec{F}_{belt}$ on the passenger is $\frac{1}{2}$ of the force on the driver, i.e.

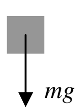
$\vec{F}_{belt} \approx \frac{1}{2} \times {}^{-}2.67 \times 10^2 = {}^{-}1.3 \times 10^2$ N.

(c) Air has a higher density than helium. The air rushes to the front when the truck slows down and forces the helium balloon to the back of the cabin.

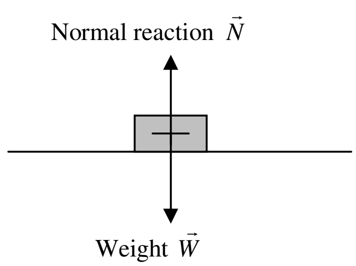
Net force F_{net} on an object

When an *m* kg object falls (assuming that there is no air resistance) near the surface of the earth, the only force on it is the force of gravity $m\vec{g}$, where $\vec{g}$ is the gravitational field strength 9.8 Nkg^{-1} towards the centre of the Earth.

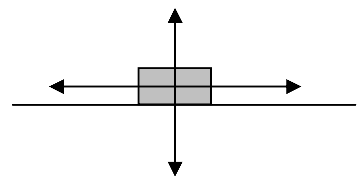
The force of gravity is also known as the **weight** of the object. It is the only force on the object, $\therefore \vec{F}_{net} = m\vec{g}$ newtons downward.



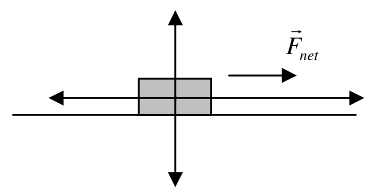
When the object rests on a horizontal surface, the surface supports the object by exerting a vertically upward force on it. This upward force has the same magnitude as the downward force of gravity, therefore the net force on the object $F_{net} = 0$. This explains why it remains at rest according to Newton's first law. The upward force is perpendicular to the surface and it is called the **normal reaction** of the surface on the object.



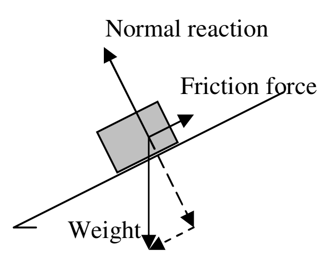
If there is a small pulling force and the surface is rough, friction exists and it prevents the object from sliding. In this case, the pulling force and the friction are equal but opposite. The vector sum of the four forces, i.e. the net force is still zero.



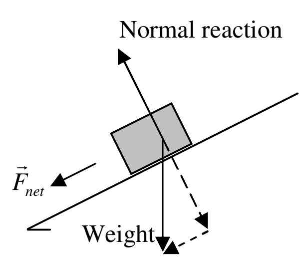
As long as the object remains at rest, friction is always equal in magnitude to the pulling (pushing) force. However, there is an upper limit to the amount of friction.
If the pulling force exceeds the maximum friction, the net force is no longer zero and causes a change in motion, i.e. acceleration, according to Newton's second law.



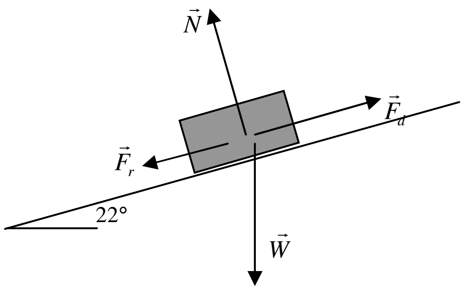
On an **inclined plane** an object at rest has zero net force acting on it.



If there is no friction (or very little friction), the vector sum of normal reaction and weight, i.e. the net force, is not zero and the object slides down the slope with increasing speed.



Example 1 A 1200-kg car travels uphill at *constant* speed along a road which inclines at an angle of 22° with the horizontal. Air resistance and rolling friction total 50 N oppose its motion. Analyse the forces acting on the car and determine the driving force.

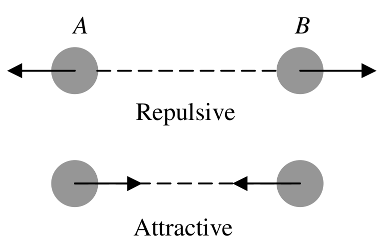


Since the car travels at constant velocity (zero acceleration), $\therefore \vec{F}_{net} = \vec{F}_d + \vec{F}_r + \vec{W} + \vec{N} = \vec{0}$, and therefore the component along the inclined plane (road) is zero. Choose uphill as the positive direction.

$^{+}F_d + {}^{-}F_r + {}^{-}W\sin\theta = 0$, $F_d = F_r + W\sin\theta$,

$F_d = 50 + 1200 \times 9.8 \times \sin 22^\circ \approx 4.5 \times 10^3$ N.

Newton's third law: Two interacting objects, *A* and *B*, exert a force on each other, i.e. *A* exerts a force $\vec{F}_B$ on *B* and *B* exerts a force $\vec{F}_A$ on *A*. $\vec{F}_A$ and $\vec{F}_B$ are equal in magnitude but opposite in direction. Usually one is called action, the other reaction.



$$\vec{F}_B = -\vec{F}_A$$

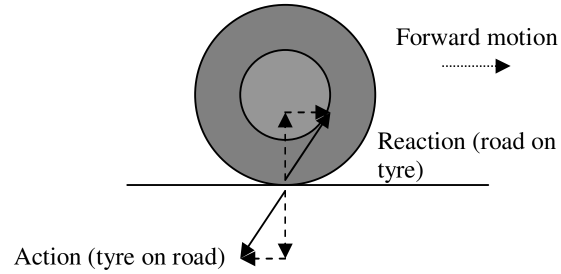
In Newton's third law there are always two objects and two forces involved. One force is on one object and the second force on the other. In Newton's second law there can be one or more forces involved and they all act on the same object.

Only forces that act on the same object can be added to give the net force. It is meaningless to add the two forces in Newton's third law and say the net force is zero.

$\vec{F}_A + \vec{F}_B = \vec{0}$, this addition is undefined.

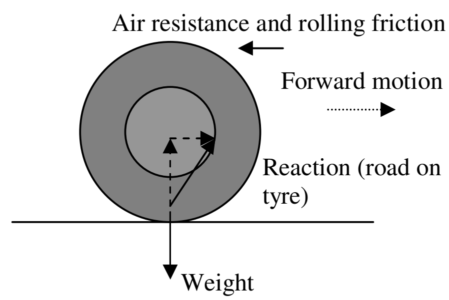
Example 1 A car travels at constant velocity on a horizontal road. Analyse the forces in Newton's third law between the tyres and the road surface. Analyse the forces in Newton's second law in relation to the motion of the car.

Force diagram showing the forces in Newton's third law:



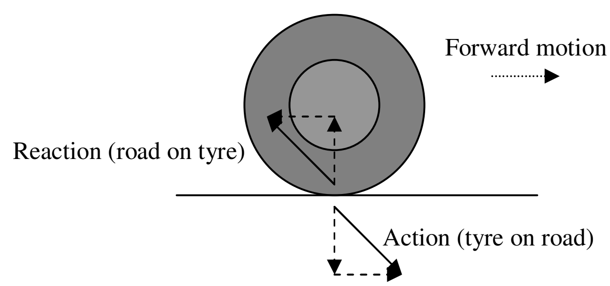
The reaction force has two components: the force of friction and the normal reaction of the road on the tyre.

Force diagram showing the forces on the car (represented by the tyre) in Newton's second law:

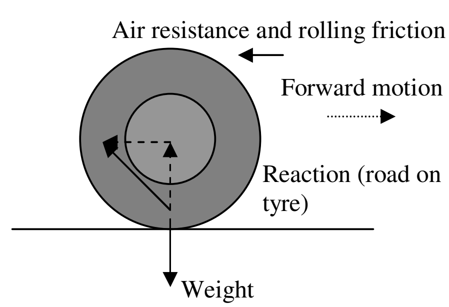


Example 2 A car slows down on a horizontal road whilst brakes are applied. Analyse the forces in Newton's third law between the tyres and the road surface. Analyse the forces in Newton's second law in relation to the motion of the car.

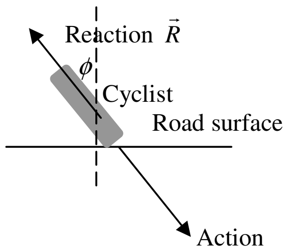
Force diagram showing the forces in Newton's third law:



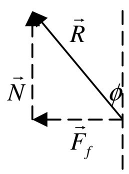
Force diagram showing the forces on the car (represented by the tyre) in Newton's second law:



Example 3 As a motorcyclist manoeuvres around a bend, she naturally slopes inwards instead of upright. The cyclist exerts a force (action) on the road and the road exerts a force (reaction) on the cyclist. These are the two forces in Newton's third law.

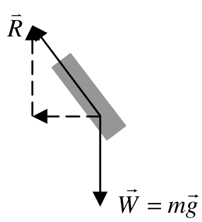


This reaction force $\vec{R}$ has two components. The vertical component is the normal reaction $\vec{N}$, and the horizontal component is the force of friction $\vec{F}_f$.



$N = R\cos\phi$, $F_f = R\sin\phi$ and $\tan\phi = \dfrac{F_f}{N}$.

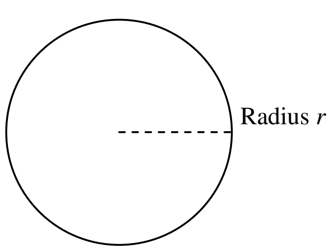
When the motion of the cyclist is analysed using Newton's second law, only forces on the cyclist are included. They are weight $\vec{W}$ and reaction force $\vec{R}$. The action force is not involved.



The net force on the cyclist is given by the horizontal component $\vec{F}_f$ because $\vec{W}$ and the vertical component of $\vec{R}$ add to zero. This net force changes the motion (direction) of the cyclist.

Uniform circular motion

When an object travels at **constant speed** in a circle, its motion is described as **uniform** circular motion.



The time for it to complete one revolution is the period T of the motion. The number of revolutions completed in a unit time is the frequency f.

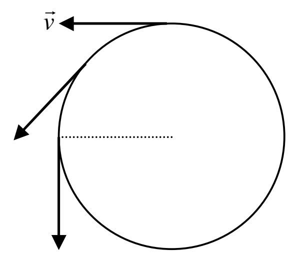
$$f = \frac{1}{T}$$

The speed v of the object can be calculated by

$$speed = \frac{dis\tan cetravelled}{timetaken}.$$

Other relationships are: $v = \dfrac{2\pi r}{T}$, $v = 2\pi rf$.

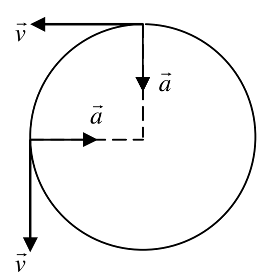
Although the speed is constant, the velocity is not because the direction of motion changes continuously as the object moves in a circle. The velocity vector $\vec{v}$ is tangential to the path.



$\therefore$ the object has an acceleration $\vec{a}$. The magnitude of $\vec{a}$ is given by

$$a = \frac{v^2}{r}$$

and the direction of $\vec{a}$ is always towards the centre of the circle, i.e. inwards along the radius of the circular path. Thus the acceleration $\vec{a}$ is sometimes called centripetal acceleration.
Since v and r are constant, $\therefore a$ is constant but $\vec{a}$ is not because its direction changes continuously as the object changes its position in the circle.



$\vec{v}$ and $\vec{a}$ are perpendicular to each other.

If the period T (or frequency f) and r are known, then
$a = \dfrac{4\pi^2 r}{T^2}$ or $a = 4\pi^2 rf^2$.

Example 1 A racing car completes 5 lapses of a round race track of radius 250 m in 5.35 min.
(a) Determine the period in hours.
(b) Determine the frequency in lapses per hour.
(c) Determine the average speed in kmh^{-1}.
(d) What is the average velocity in kmh^{-1}?

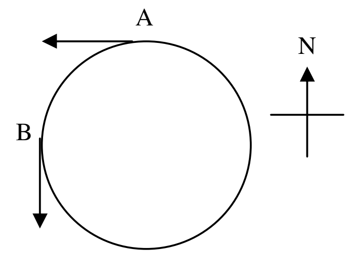
(a) $T = \frac{5.35}{5} = 1.07 \text{ min} = 0.018 \text{ h}$

(b) $f = \frac{1}{T} = 56$ lapses per hour

(c) $v_{av} = \frac{2\pi r}{T} = 88.1 \text{kmh}^{-1}$

(d) $\vec{v}_{av} = \frac{s}{\Delta t} = 0$

Example 2 A cyclist travels around a round-about of radius 6.2 m at 3.1 ms^{-1}.



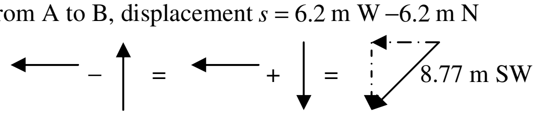
(a) Determine her accelerations at A and B.
(b) Determine her positions at A and B relative to the centre O of the round about.
(c) Calculate her average velocity from A to B.
(d) Determine her velocities at A and B.
(e) Calculate her average acceleration from A to B.

(a) $a = \frac{v^2}{r} = 1.6 \text{ms}^{-2}$.
1.6 ms^{-2} south at A, 1.6 ms^{-2} east at B.

(b) 6.2 m north of O at A, 6.2 m west of O at B.

(c) Time for one round $= T = \frac{2\pi r}{v} = 12.57$ s

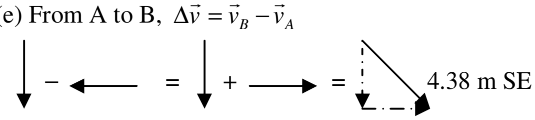
Time to go from A to B $= \frac{T}{4} = 3.14$ s
From A to B, displacement s = 6.2 m W −6.2 m N



$\vec{v}_{av} = \frac{s}{\Delta t} = 2.8 \text{ ms}^{-1}$ SW

(d) $\vec{v}_A = 3.1 \text{ms}^{-1}$ W, $\vec{v}_B = 3.1 \text{ms}^{-1}$ S

(e) From A to B, $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$



$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = 1.4 \text{ ms}^{-2}$ SE

Example 3 A 1500 kg car travels at 15 ms^{-1} around a flat horizontal bend of radius 50 m.
(a) Determine the total friction (between the tyres and the road surface) which helps the car to make the turn.
(b) If the maximum possible friction is 0.6 times the normal reaction of the road on the car, what is the maximum speed the car can have before it skids out of control?

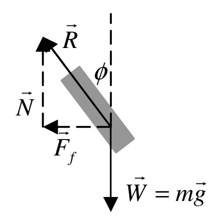
(a) $a = \frac{v^2}{r} = \frac{15^2}{50} = 4.5 \text{ ms}^{-2}$.

$F_{net} = ma$, $F_{net} = F_f = 1500 \times 4.5 = 6.75 \times 10^4$ N.
(b) Max acceleration:
$a_{\max} = \frac{F_{f,\max}}{m} = \frac{0.6 \times 1500 \times 9.8}{1500} = 5.88 \text{ ms}^{-2}$.

$a_{\max} = \frac{{v_{\max}}^2}{r}$, $v_{\max} = \sqrt{a_{\max} r} = 17 \text{ ms}^{-1}$.

Example 4 Refer to previous discussion of a motorcyclist moving around a bend. The total mass of the rider and the motorcycle is 850 kg, and the motorcycle makes an angle of 20° with the vertical. The radius of the bend is 50 m.
(a) Determine the normal reaction *N* of the road surface on the cycle.
(b) Calculate the friction F_f between the tyres and the road.
(c) Determine the acceleration *a* of the motorcycle.
(d) Calculate the speed *v* of the motorcycle.
(e) What is the speed if another person who is 10 kg heavier rides the same motorcycle under the same conditions?
(f) Show the relationship between angle ϕ with the vertical, speed *v* of the motorcycle and radius *r* of the bend.

(a)



$N = W = mg = 850 \times 9.8 = 8330$ N

(b) $\frac{F_f}{N} = \tan\phi$, $F_f = N\tan\phi = 8330 \tan 20° = 3.0 \times 10^3$ N

(c) $a = \frac{F_{net}}{m} = \frac{F_f}{m} = 3.6 \text{ ms}^{-2}$

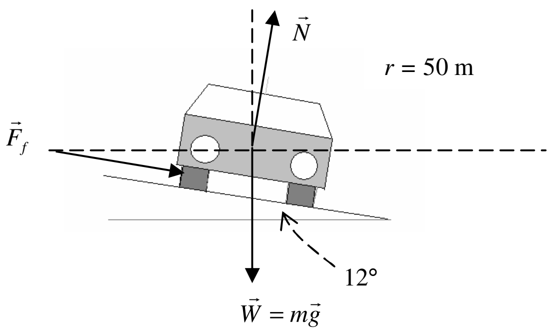
(d) $a = \frac{v^2}{r}$, $v = \sqrt{ar} = 13 \text{ ms}^{-1}$

(e) Same speed because speed around the bend is independent of mass. $N = W = mg$, $F_f = N\tan\phi = mg\tan\phi$,

$a = \frac{F_{net}}{m} = \frac{F_f}{m} = g\tan\phi$, $\therefore v = \sqrt{ar} = \sqrt{gr\tan\phi}$.

(f) $\tan\phi = \frac{v^2}{gr}$, $\phi = \tan^{-1}\left(\frac{v^2}{gr}\right)$, provided $F_{f,\max} \geq mg\tan\phi$,
i.e. as long as the friction between the tyres and the road is less than the maximum possible friction, the leaning angle ϕ increases with increasing speed *v*, and with smaller radius *r*.

Example 5 A 1500-kg car enters a turn whose radius is 50 m. The road is banked at an angle of 12°. The maximum friction force between the tyres and the road surface is 0.6 times the normal reaction force of the road on the car.
(a) Calculate the force of friction F_f on the tyres when the car travels at a constant speed of 15 ms^{-1}.
(b) Find the maximum speed $v_{\max}$ of the car before it skids out of control.
(c) Find the speed that no friction is required to make the turn. Note: This speed is called the *design speed* for the turn.
(d) State the benefits of a banked road to motorists making a turn.



(a) There are 3 forces on the car: $\vec{N}$, $\vec{W}$ and $\vec{F}_f$.
Resolve the forces into vertical and *horizontal* components because the car is in *horizontal* uniform circular motion.
Vertically: $F_{net} = 0$, $N\cos 12° - F_f \sin 12° - 1500\times 9.8 = 0$.

$N - F_f \tan 12° = \dfrac{1500\times 9.8}{\cos 12°}$, $N - 0.2126F_f = 15028$ ……(1)

Horizontally: $F_{net} = ma$, $F_{net} = \dfrac{mv^2}{r}$,

$F_f \cos 12° + N\sin 12° = \dfrac{1500\times 15^2}{50}$.

$\dfrac{F_f}{\tan 12°} + N = \dfrac{1500\times 15^2}{50\sin 12°}$, $4.7046F_f + N = 32466$ ……(2)

(2) − (1), $4.9172F_f = 17437$, $F_f = 3.5\times 10^3$ N.
Note: Compare with example 3(a).

(b) Maximum friction force $F_{f,\max} = 0.6N$.
Vertically: $F_{net} = 0$, $N\cos 12° - F_f \sin 12° - 1500\times 9.8 = 0$,
$N\cos 12° - 0.6N\sin 12° - 1500\times 9.8 = 0$, $N = 17225$.

Horizontally: $F_f \cos 12° + N\sin 12° = \dfrac{1500\times v^2}{50}$,

$0.6N\cos 12° + N\sin 12° = \dfrac{1500\times v^2}{50}$, $\therefore v = 21.4$ ms^{-1} is the maximum speed. Note: Compare with example 3(b).

(c) $F_f = 0$
Vertically: $F_{net} = 0$, $N\cos 12° = 1500\times 9.8$ ……(1)

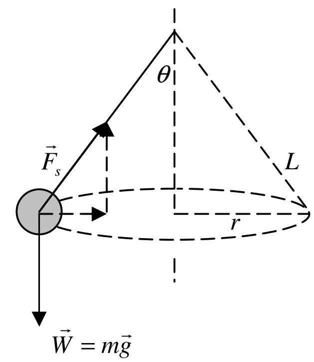
Horizontally: $N\sin 12° = \dfrac{1500\times v^2}{50}$ ……(2)

(2)/(1) : $\tan 12° = \dfrac{v^2}{9.8\times 50}$, $v = 10.2$ ms^{-1}.

Note: $v = \sqrt{gr\tan\theta}$ gives the design speed.

(d) A banked road (i) creates an extra horizontal component of normal reaction force and thus the centripetal force required to make a turn is less reliant on friction force as its source,
(ii) increases the normal reaction force on the car during the turn and thus increases the maximum friction force between the tyres and the road. Hence the maximum allowable speed is increased.

Example 6 **Conical pendulum**: A particle of mass *m* kg, suspended by a light string of length *L* m, revolves in a horizontal circle with the string making an angle of $\theta°$ with the vertical.
(a) Find the tension F_s in the string in terms of *m* and θ.
(b) Show that the speed *v* of the particle is independent of its mass *m*.
(c) Show that the period *T* of revolution of the particle is also independent of its mass *m*.



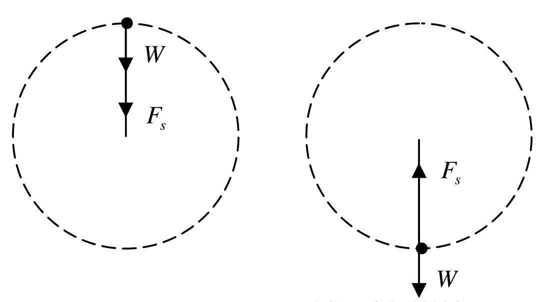
(a) Vertically: $F_s\cos\theta - mg = 0$, $\therefore F_s = \dfrac{mg}{\cos\theta}$.

(b) Horizontally: $F_s\sin\theta = \dfrac{mv^2}{r} = \dfrac{mv^2}{L\sin\theta}$,

$\therefore \dfrac{mg}{\cos\theta}\sin\theta = \dfrac{mv^2}{L\sin\theta}$, $\therefore v = \sin\theta\sqrt{\dfrac{Lg}{\cos\theta}}$, independent of *m*.

(c) $v = \dfrac{2\pi r}{T}$, $\therefore \sin\theta\sqrt{\dfrac{Lg}{\cos\theta}} = \dfrac{2\pi L\sin\theta}{T}$, $\therefore T = 2\pi\sqrt{\dfrac{L\cos\theta}{g}}$, independent of *m*.

Example 7 A 1.0 kg particle at the end of a 1.0 m light string is made to revolve in a vertical circle. Its speed is 4.00 ms^{-1} at the highest point and 7.43 ms^{-1} at the lowest point. Find the tension F_s in the string at each location.



Take upward as the positive direction.

At the highest point: $\vec{F}_{net} = m\vec{a}$, ${}^{-}F_s + {}^{-}(mg) = m\left(\frac{{}^{-}v^2}{r}\right)$,

${}^{-}F_s + {}^{-}9.8 = {}^{-}16$, $F_s = 6.2$ N.

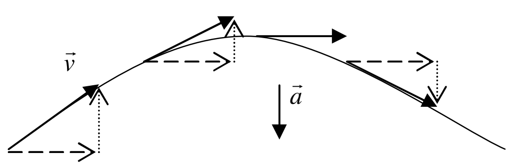
At the lowest point: $\vec{F}_{net} = m\vec{a}$, ${}^{+}F_s + {}^{-}(mg) = m\left(\frac{{}^{+}v^2}{r}\right)$,

${}^{+}F_s + {}^{-}9.8 = {}^{+}55.2$, $F_s = 65$ N.

Motion of projectiles near Earth's surface

Within a short distance the force of gravity on an object can be considered as constant with a magnitude of approximately 9.8 N per kilogram of the object and its direction is vertically downward. This constant force gives the object a constant downward acceleration of 9.8 ms^{-2} .

In analysing projectile motion, usually it is resolved into two components, namely vertical and horizontal, and each component is analysed separately as one dimensional motion.



The horizontal component of the velocity vector is unaffected by gravity and it remains constant throughout the flight.

The vertical component changes continuously while the projectile is in flight because it is affected by gravity. It has a constant downward acceleration of 9.8 ms^{-2}. Because the vertical component is one-dimensional and has a constant acceleration, therefore, the five equations of motion in a straight line under constant acceleration stated below are applicable.

$$v = u + at$$

$$s = \frac{1}{2}(u + v)t$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

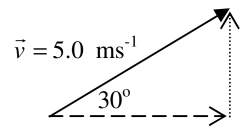
$$v^2 = u^2 + 2as$$

Each equation involved four of the following five quantities:

u initial velocity, i.e. velocity at $t = 0$
v final velocity
a the constant acceleration
s displacement (not distance travelled nor position)
t at time t.

For the horizontal component, $s = ut$ is the only equation because the acceleration is zero.

Resolving a velocity vector into horizontal and vertical components
E.g.



The vertical component of $\vec{v} = 5.0\sin 30° = {}^{+}2.5$ ms^{-1}.

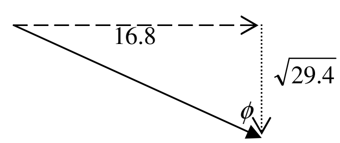
The horizontal component of $\vec{v} = 5.0\cos 30° = {}^{+}4.3$ ms^{-1}.

Example 1 Due to brake failure a car hit a barrier and came to a sudden stop. A surfboard loosely fastened to the roof was projected forward with an initial speed of 16.8 ms^{-1} because of its momentum. The roof of the car is 1.5 m above the ground.
(a) Calculate the time of flight of the surfboard.
(b) How far away from the car did it hit the ground?
(c) At what speed did it hit the ground?
(d) At what angle with the vertical did it hit the ground?

(a) Vertically: $u = 0$, $s = {}^{-}1.5$, $a = {}^{-}9.8$. Use $s = ut + \frac{1}{2}at^2$ to find *t*. ${}^{-}1.5 = \frac{1}{2}({}^{-}9.8)t^2$, $t = 0.55$ s.

(b) Horizontally: $u = {}^{+}16.8$, $t = 0.55$,
$s = ut = {}^{+}16.8 \times 0.55 = {}^{+}9.3$ m.

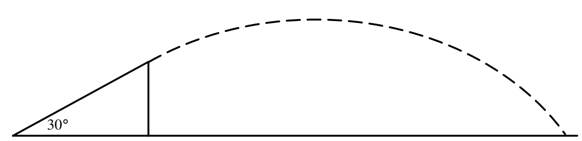
(c) Vertically: $u = 0$, $s = {}^{-}1.5$, $a = {}^{-}9.8$. Use $v^2 = u^2 + 2as$ to find v^2 . $v^2 = 29.4$.



Speed $= \sqrt{16.8^2 + 29.4} \approx 18$ ms^{-1}.

(d) $\tan\phi = \frac{16.8}{\sqrt{29.4}}$, $\phi = 72°$.

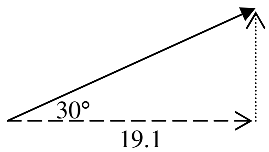
Example 2 A daredevil rides a motorcycle up a ramp inclined at 30° with the horizontal. She clears a distance of 50.4 m from the base of the ramp after 2.64 s in the air.



(a) Find the horizontal component of her velocity in the air.
(b) Find the vertical component of her velocity at take off.
(c) Calculate her velocity at take off.
(d) How high is the end of the ramp above the ground?
(e) Find the maximum height of the cycle above the ground?

(a) Horizontally: $s = {}^{+}50.4$, $t = 2.64$, $u = \frac{s}{t} = {}^{+}19.1\ \text{ms}^{-1}$.

(b)



Vertical component of velocity at take off $= 19.1\tan 30° = 11.0$.

(c) Velocity at take off $= \sqrt{19.1^2 + 11.0^2} = 22.0\ \text{ms}^{-1}$ at 30° with the horizontal.

(d) Vertically: $u = {}^{+}11.0$, $a = {}^{-}9.8$, $t = 2.64$. Use $s = ut + \frac{1}{2}at^2$ to find *s*.

$s = {}^{+}11.0 \times 2.64 + \frac{1}{2}\left({}^{-}9.8\right)2.64^2 = {}^{-}5.11\ \text{m}$.

The height of the ramp is 5.11 m.

(e) Vertically: $u = {}^{+}11.0$, $a = {}^{-}9.8$, $v = 0$. Use $v^2 = u^2 + 2as$ to find *s*. $0 = 11.0^2 + 2\left({}^{-}9.8\right)s$, $s = {}^{+}6.17\ \text{m}$.

Maximum height above the ground $= 5.11 + 6.17 = 11.3\ \text{m}$.

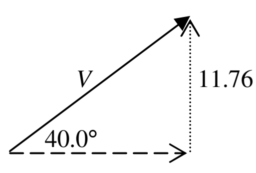
Example 3 A stone is catapulted into the air at an angle of 40.0° above level ground. It reaches maximum height 1.20 s later.

(a) What is its acceleration at the highest point?
(b) What is its average acceleration in the first 1.20 s?
(c) Find the speed of projection of the stone.
(d) Find its average velocity in the first 1.20 s.
(e) Find its average velocity in the first 2.40 s.

(a) The acceleration is constant throughout the motion, 9.8 ms^{-2} downward.

(b) Since the acceleration is constant, $\therefore \vec{a}_{av} = 9.8\ \text{ms}^{-2}$ downward.

(c) Vertically: $v = 0$, $a = {}^{-}9.8$, $t = 1.20$. Use $v = u + at$ to find *u*. $0 = u + {}^{-}9.8 \times 1.20$, $u = {}^{+}11.76$.

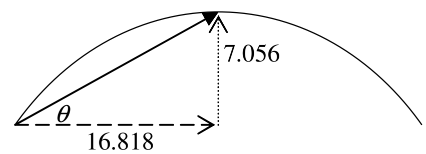


Speed of projection $V = \frac{11.76}{\sin 40.0°} \approx 18.3\ \text{ms}^{-1}$.

Horizontal component of $\vec{V} = 18.3\cos 40.0° = 14.015\ \text{ms}^{-1}$ (for parts d and e).

(d) Vertically: $v = 0$, $a = {}^{-}9.8$, $t = 1.20$. Use $s = vt - \frac{1}{2}at^2$ to find *s*. $s = -\frac{1}{2}\left({}^{-}9.8\right)1.20^2 = {}^{+}7.056\ \text{m}$

Horizontally: $u = {}^{+}14.015$, $t = 1.20$, $s = ut = {}^{+}16.818$.



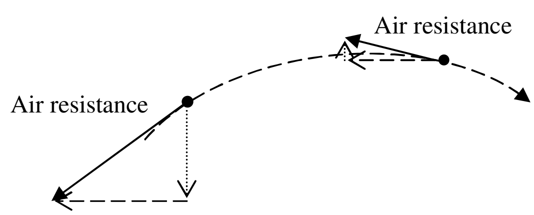
Displacement in the first 1.20 s $= \sqrt{16.818^2 + 7.056^2} \approx 18.2\ \text{m}$ at $\theta = \tan^{-1}\left(\frac{7.056}{16.818}\right) \approx 22.8°$ above the level ground.

$\vec{v}_{av} = \frac{s}{\Delta t} = \frac{18.2}{1.20} \approx 15.2\ \text{ms}^{-1}$ at 22.8° above the level ground.

(e) The displacement of the stone in the first 2.40 s is horizontal. The average velocity is given by the horizontal component of its velocity (which remains constant throughout the motion), i.e. 14.0 ms^{-1}.

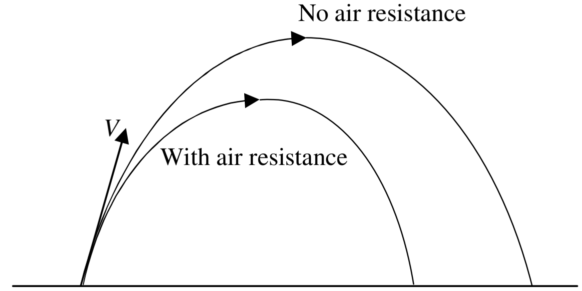
Effects of air resistance on motion of projectiles

Air resistance exerts a force on a projectile opposite in direction to its motion through the air. This air resistance force affects both components of the projectile motion.



The acceleration of the projectile is now due to gravity and air resistance, and the vertical acceleration and horizontal velocity cannot be considered as constant. Furthermore the vertical and horizontal components of the motion are no longer independent and cannot be analysed separately. The overall effects of air resistance on motion of a projectile are:

(i) reduction in both height and range
(ii) asymmetric flight path.



Newton's second law in terms of impulse and momentum (in one dimension)

Only constant net force is considered in order to simplify the situation, otherwise calculus is required for the analysis.

Consider an object of mass m kg experiencing a net force F_{net} newtons over a time interval Δt seconds. The net force changes its velocity from u ms^{-1} to v during this time interval.

Since the net force is constant, the acceleration is constant and given by $\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}-\vec{u}}{\Delta t}$.

Substitute $\vec{a} = \frac{\vec{v}-\vec{u}}{\Delta t}$ in $\vec{F}_{net} = m\vec{a}$ to obtain

$$\vec{F}_{net}\Delta t = m\vec{v} - m\vec{u}.$$

The quantity on the left $\vec{F}_{net}\Delta t$ is called impulse $\vec{I}$ given to the object by the net force. $\vec{I}$ is measured in Ns and is in the same direction as the net force, i.e. $\vec{I} = \vec{F}_{net}\Delta t$

The two quantities on the right are called final momentum $\vec{p}_f$ and initial momentum $\vec{p}_i$ respectively. The difference is called the change in momentum $\Delta\vec{p}$. Momentum is measured in kg ms^{-1} and it is a vector in the same direction as velocity.

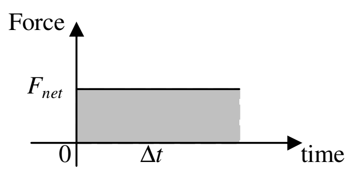
$\therefore \vec{p}_i = m\vec{u}$, $\vec{p}_f = m\vec{v}$ and $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = m\vec{v} - m\vec{u}$

Note that the unit for impulse Ns is equivalent to that for momentum kg ms^{-1}.

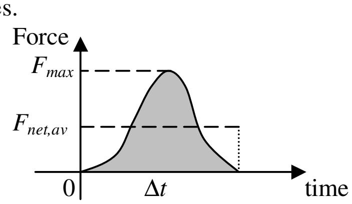
Now we have Newton's second law in terms of impulse and momentum, i.e. $\vec{I} = \Delta\vec{p}$ instead of $\vec{F}_{net} = m\vec{a}$.

Note that $\vec{I}$ and $\Delta\vec{p}$ are in the same direction.

The area under a force-time graph represents impulse (which is also equal to the change in momentum).



For non-constant force, e.g. force during a collision, the same idea applies.



The area under the curve equals the area under the line $F_{net,av}$,

$\therefore \vec{F}_{net,av}\Delta t = \Delta\vec{p}$ or $\vec{F}_{net,av} = \frac{\Delta\vec{p}}{\Delta t}$.

$$\therefore F_{net,av} \propto \frac{1}{\Delta t} \text{ for constant } \Delta p.$$

If Δt is smaller, $F_{net,av}$ becomes larger. This explains why, for example, a person hitting an airbag sustains lesser injuries than a person hitting a strong rigid windscreen in a car accident. Both persons (of the same mass) have the same change in momentum, but the person hitting the windscreen has a shorter impact time and therefore experiences a stronger impact force than the other person hitting the airbag that helps to extend the stopping time.

Example1 In a laboratory test a car without a crumple zone hits a strong barrier and comes to a halt. Two crash test dummies of the same mass 60 kg are placed at the front without seatbelts on. One of the dummies hits the windscreen and the other hits an airbag at 50 km h^{-1}. The measured stopping times are 0.003 s and 0.036 s respectively. Estimate the average net force on each dummy.

The following calculation is an oversimplification of the real situation because different parts of the dummies hit the car interior at different places and at different times. Assume that the effective mass hitting the windscreen or airbag is 7 kg.

50 km h^{-1} = 13.9 ms^{-1}.

$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 0 - 7 \times {}^{+}13.9 = {}^{-}97$ kg ms^{-1}.

Windscreen: $\vec{F}_{net,av} = \frac{\Delta\vec{p}}{\Delta t} = \frac{{}^{-}97}{0.003} \approx {}^{-}3\times10^4$ N.

Airbag: $\vec{F}_{net,av} = \frac{{}^{-}97}{0.036} \approx {}^{-}3\times10^3$ N.

Modern cars have a crumple zone at the front end. This effectively lengthens the stopping time of a car. If seatbelt is fastened, the occupant will have the same motion as the car and therefore about the same stopping time (0.1-0.2 s) as the car. Hence the average force is smaller.

Example 2 In another crash test, a 60-kg dummy is fastened to the front seat with a standard seatbelt. The car has a crumple zone at the front end and hits a strong barrier at 50 km h^{-1}. It comes to a halt. The recorded stopping time of the driver compartment is 0.15 s. Estimate the average net force on the dummy exerted by the seatbelt.

$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 0 - 60 \times {}^{+}13.9 \approx {}^{-}830$ kg ms^{-1}.

$\vec{F}_{net,av} = \frac{\Delta\vec{p}}{\Delta t} = \frac{{}^{-}830}{0.15} \approx {}^{-}6\times10^3$ N.

Note: The forces calculated above must be considered together with the contact areas. The quantity *pressure* $\left(=\frac{F}{A}\right)$ is used to compare the effects of these forces.

Windscreen: Estimated contact area $\approx 0.01\,\text{m}^2$,
pressure $\approx \frac{3\times10^4}{0.01} = 3\times10^6\,\text{Nm}^{-2}$.

Airbag: Estimated contact area $\approx 0.03\,\text{m}^2$,
pressure $\approx \frac{3\times10^3}{0.03} = 1\times10^5\,\text{Nm}^{-2}$.

Seatbelt: Estimated contact area $\approx 0.05\,\text{m}^2$,
pressure $\approx \frac{6\times10^3}{0.05} \approx 1\times10^5\,\text{Nm}^{-2}$.

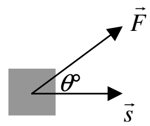
The pressure exerted by the windscreen is about 30 times that by the airbag or seatbelt.

Work done by a constant force

Work done W by a constant force $\vec{F}$ on an object over a displacement $\vec{s}$ is defined as $W = Fs$ if $\vec{F}$ and $\vec{s}$ are in the same direction. W in this situation is a positive amount of work and the object gains energy.

If $\vec{F}$ and $\vec{s}$ are in opposite directions. W is a negative amount of work, $W = -Fs$ and the object loses energy.

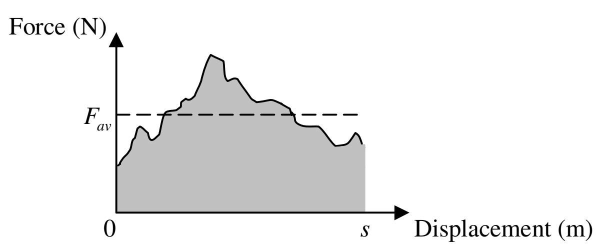
If $\vec{F}$ and $\vec{s}$ are at an angle $\theta°$, the work done is given by $W = Fs\cos\theta°$.



Work and energy are scalar quantities and they are measured in joules(J).

Work done by a variable force

When force $\vec{F}$ on an object changes with its displacement $\vec{s}$, area under the force-displacement graph represents work done by $\vec{F}$ if $\vec{F}$ and $\vec{s}$ are in the same direction. Estimate the area if it cannot be determined by simple calculation.



The estimated area $\approx F_{av}s$.

Example 3 A crate is pulled along a rough level surface with a rope at an angle of 25° above the horizontal. The tension in the rope is 100 N and friction force against motion is 80 N.
(a) Find the work done by the pulling force when the crate moves a distance of 1.2 m.
(b) Find the work done by the friction force over the 1.2 m.
(c) Find the work done by the normal reaction on the crate.
(d) Find the work done by the net force over the 1.2 m.

(a) $W = Fs\cos\theta° = 100\times1.2\times\cos25° \approx 110$ J (108.8 J)

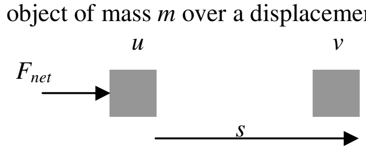
(b) $W = -Fs = -80\times1.2 = -96$ J. The negative sign indicates that energy is taken out of the system.

(c) $W = Fs\cos\theta° = N\times1.2\times\cos90° = 0$, where N is the normal reaction force on the crate.

(d) $\vec{F}_{net} = {}^{+}100\cos25° + {}^{-}80 = {}^{+}10.63$ N
$W = F_{net}s = 10.63\times1.2 \approx 13$ J
Alternatively, $W = 108.8 - 96 \approx 13$ J

Newton's second law in terms of work and energy

To simplify the situation consider a constant net force F_{net} acting on an object of mass m over a displacement s.



Since F_{net} is constant, acceleration a is also constant.

$$\therefore v^2 = u^2 + 2as,\ \therefore a = \frac{v^2 - u^2}{2s}.$$

Newton's second law: $F_{net} = ma$, $\therefore F_{net}s = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

The quantity $F_{net}s$ is the work W done by F_{net} on the object.

The quantities $\frac{1}{2}mv^2$ and $\frac{1}{2}mu^2$ are defined as the final kinetic energy and the initial kinetic energy respectively. Their difference is called the change in kinetic energy ΔE_k.

Therefore Newton's second law can be stated as work done by *net force* equals *change* in kinetic energy of the object.

$$W = \Delta E_k.$$

This idea of 'work done by the net force on an object equals its change in kinetic energy' can also explain why crumple zone is an important safety feature of a car. The crumple zone extends the stopping distance s of an occupant wearing a seatbelt and thus lowers the force on the person.

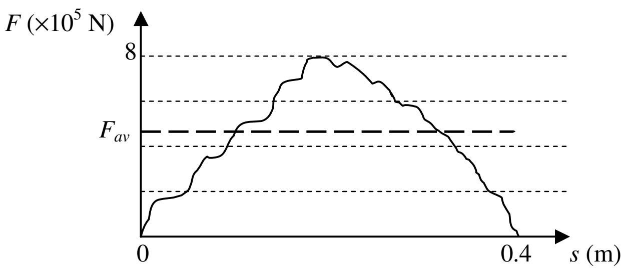
$F_{net,av} = \frac{|\Delta E_k|}{s}$, and ΔE_k of the occupant is constant under the same condition in a crash, $\therefore F_{net} \propto \frac{1}{s}$.

Example 4 In example 2 on page 9, the crumple zone is crushed by a distance of 1 m and the stopping time is unknown. Estimate the average net force exerted by the seatbelt on the dummy.

$$\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = 0 - \frac{1}{2}\times 60\times 13.9^2 \approx -6000\,\text{J}$$

$$F_{net,av} = \frac{|\Delta E_k|}{s} = \frac{6000}{1} = 6\times 10^3\,\text{N}.$$

Example 5 The following graph shows the force on a 1-tonne car versus displacement of its centre of mass when it hits a barrier.
(a) Find the average force in stopping the car.
(b) Find the speed of the car just before it hits the barrier.



(a) Draw a horizontal dotted line representing F_{av} so that the area under it equals the area under the curve. Read from the vertical scale, $F_{av} \approx 4.7\times 10^5$ N.

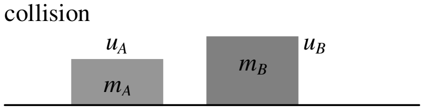
(b) $|\Delta E_k| = W$, $\left|0 - \frac{1}{2}mu^2\right| = F_{av}s$,

$\frac{1}{2}\times 1000\times u^2 = 4.7\times 10^5\times 0.4$, $u \approx 19\,\text{ms}^{-1}$.

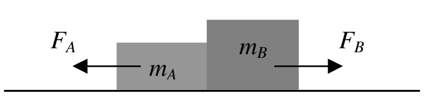
Conservation of momentum – a consequence of Newton's third law

Consider a system of two objects A and B sliding along a smooth surface in a straight line, approaching and colliding with each other head on. The system can be considered *isolated* because the net external force on it is zero. Friction is negligible and the force of gravity is balanced by the normal reaction. Within the system there is a mutual interacting force between the two objects during collision.

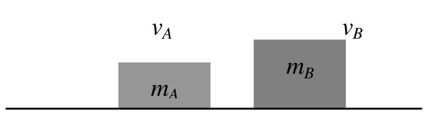
Before collision



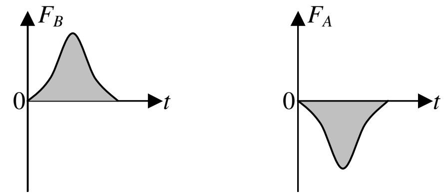
During collision



After collision



During collision, according to Newton's third law, $F_A = -F_B$.



The impulse (area 'under' the force-time graph) given to A by B is equal but opposite to that given to B by A.

Therefore, according to Newton's second law, the change in momentum of A is equal but opposite to that of B.

$$m_Bv_B - m_Bu_B = -(m_Av_A - m_Au_A)$$

$$\therefore\ m_Av_A + m_Bv_B = m_Au_A + m_Bu_B$$

i.e. the total momentum after collision is equal to the total momentum before collision. Even during collision, total momentum is also conserved. This is known as the Law of conservation of momentum. The total momentum remains constant for an isolated system.

In real life situations, e.g. a two-car head on collision, only the total momenta *immediately* before and after the collision are considered otherwise the effects of friction are too great to be ignored and the system can no longer be treated as isolated.

Example 1 Due to signal malfunction, a train of mass *M* kg travelling at 30 km h^{-1} collides with another of mass 2.5*M* kg travelling in the opposite direction at 35 km h^{-1}. The two trains lock together after the collision.
(a) Determine the velocity of the locked trains immediately after the collision.
(b) In terms of *M*, calculate the total kinetic energy of the two trains immediately before the collision.
(c) Compare the answer in (b) with the total kinetic energy immediately after the collision. Discuss the difference.

(a) It is not necessary to change km h^{-1} to ms^{-1} in this part.
Total momentum just after = total momentum just before
$(M + 2.5M)\vec{V} = M\times{}^{+}30 + 2.5M\times{}^{-}35$
$3.5\vec{V} = {}^{-}57.5$, $\therefore \vec{V} = {}^{-}16.4$ km h^{-1}.

(b) Before collision:
30 km h^{-1} = 8.33 ms^{-1}, 35 km h^{-1} = 9.72 ms^{-1}.
Total kinetic energy $= \frac{1}{2}\times M\times 8.33^2 + \frac{1}{2}\times 2.5M\times 9.72^2$
$\approx 153M$ J.

(c) After collision: $\vec{V} = {}^{-}16.4$ km h^{-1} = ${}^{-}4.56$ ms^{-1}.
Total kinetic energy $= \frac{1}{2}\times(M + 2.5M)\times 4.56^2 \approx 36M$ J.

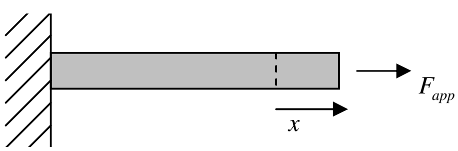
Some of the kinetic energy before collision is transformed to heat, sound and strain energies after collision.

The type of collisions discussed in the last example is described as *inelastic collision*. In an inelastic collision, the total kinetic energy after collision is different from (note: not always less than, e.g. when a rifle is fired) the total kinetic energy before collision, otherwise it is called *elastic*. In both cases, the total momentum is conserved.

Example 2 Two identical train carriages A and B travel on the same tracks in the same direction. Carriage A travels at 2 ms^{-1} and rolls into carriage B travelling at 1 ms^{-1}. Assume that it is an elastic collision. Calculate the velocities of A and B immediately after the collision.

Let v_A and v_B be the velocities of A and B after collision respectively.
Conservation of momentum:
$mv_A + mv_B = m\times 2 + m\times 1$, $\therefore v_A + v_B = 3$ ……(1)
Elastic collision:
$\frac{1}{2}mv_A^{\ 2} + \frac{1}{2}mv_B^{\ 2} = \frac{1}{2}m\times 2^2 + \frac{1}{2}m\times 1^2$,
$\therefore v_A^{\ 2} + v_B^{\ 2} = 5$ ……(2)
Square (1), $v_A^{\ 2} + 2v_Av_B + v_B^{\ 2} = 9$ ……(3)
(3) − (2), $v_Av_B = 2$, $v_B = \frac{2}{v_A}$ ……(4)
Substitute (4) in (1), $v_A + \frac{2}{v_A} = 3$, $\therefore v_A^{\ 2} - 3v_A + 2 = 0$.
Hence $v_A = 1$ ms^{-1} and $v_B = 2$ ms^{-1}.

Hooke's law and elastic potential energy

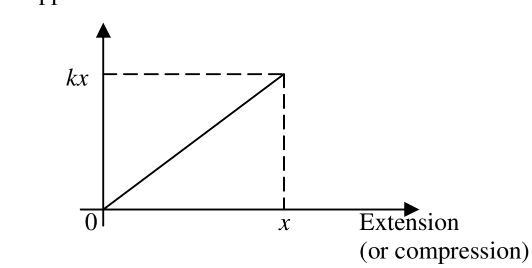


When a material (or spring) is compressed or stretched, the applied force F_{app} is directly proportional to the amount x compressed or stretched provided it is not overdone. This is known as Hooke's law.

$F_{app} \propto x$, $\therefore F_{app} = kx$ where k is called the force constant of the material and has the unit Nm^{-1} if F_{app} and x are measured in N and m respectively.

Hooke's law can also be expressed in terms of the force F exerted by a material (or spring).

$$F = -kx$$



The value of k is given by the gradient of the F-x graph.

The amount of work done (J) by the applied force or the amount of elastic strain energy (J) stored in the material is given by the area under the F-x graph.

$$W = E_p = \frac{1}{2}kx^2$$

When the amount of compression or extension changes from x_a to x_b, the additional work done or the increase in elastic strain energy is

$$W = \Delta E_p = \frac{1}{2}kx_b^2 - \frac{1}{2}kx_a^2$$

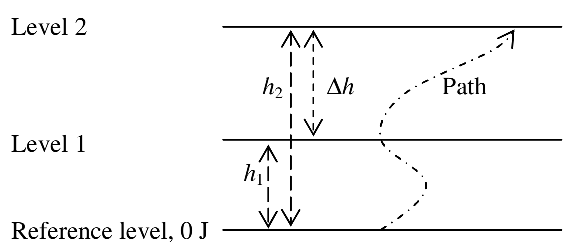
Gravitational potential energy

To measure gravitational potential energy at (near) the surface of the earth, a level (usually the lowest but not necessarily) is chosen as the reference level and assigned a value of 0 J. Above this reference level the gravitational potential energy is a positive value, below it is negative.

Consider two levels above the reference level. Level 1 is h_1 metres above the reference level and level 2 is h_2 metres above the reference level. To raise a m-kg object at constant speed, a force equals to the weight of the object, i.e. mg newtons, is applied. The work done to raise the object from the reference level to level 1 is mgh_1 J, and from the reference level to level 2 is mgh_2 J. The object's gravitational potential energy at each level is mgh_1 J and mgh_2 J respectively. Gravitational potential energy depends on the vertical distance only and not the path from one level to another.

The difference (or change) in gravitational potential energy when the object is raised from level 1 to level 2 is

$$\Delta E_p = mgh_2 - mgh_1 = mg(h_2 - h_1) = mg\Delta h$$



The law of conservation of energy

Energy changes from one form to another and can be transferred from one object to another during interaction. Work is done in the transformation or transfer of energy, e.g. in a car collision the head of the driver presses against the airbag. Work is done by the head to compress the airbag. In doing so the kinetic energy of the head changes to strain energy E_s in deforming the airbag and some heat is also generated.

$$E_k \xrightarrow{\text{Work}} E_s + \text{heat}$$

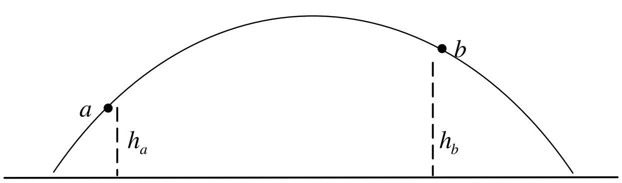
To a good approximation the total amount of energy ($E_k + E_s$ + heat) at any time during the collision of the head with the airbag is constant. This example is an approximation because the system of objects (the head and the airbag) is not isolated for obvious reasons.

For an isolated system, the total amount of energy is constant. This is known as the **law of conservation of energy**.

Conservation of energy and projectile motion

The system consists of the projectile and the earth. It can be considered as isolated, and there are two types of energy involved. They are the kinetic energies of the projectile and the earth, and the gravitational potential energy between them.

The mass of the earth is very much greater than the mass of the projectile. This results in practically zero velocity for the earth relative to the centre of mass of the system. Therefore when one applies the law of conservation of energy to projectile motions, it is simpler to take the total amount of energy as the sum of the gravitational potential energy and the kinetic energy of the projectile, and it is constant at different stages of the motion.



$$mgh_a + \frac{1}{2}mv_a^2 = mgh_b + \frac{1}{2}mv_b^2 \qquad (1)$$

or

$$\Delta E_k + \Delta E_p = 0 \qquad (2)$$

Equation (2) suggests that when kinetic energy increases (decreases), potential energy decreases (increases) by the same amount and the changes always add to zero.

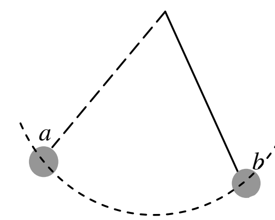
If air resistance is taken into account, heat must be added to equation (2).

Conservation of energy and vertical circular motion

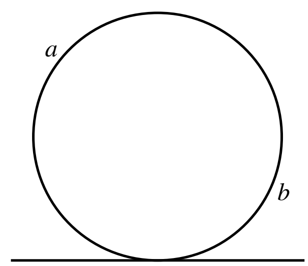
Consider the system consisting of the earth and an object moving in a vertical circle. Similar to the projectile motion in the previous section, the system can be simplified to that of an object moving in a vertical circle under the influence of gravity, without taking the earth's kinetic energy into account.

If friction is negligible, the total energy is the sum of gravitational potential energy and the kinetic energy of the object. It is constant at different stages of the motion.

Example 1 A simple pendulum.



Example 2 Roller coaster ride.



The previous two equations are also valid for these situations.

$$mgh_a + \frac{1}{2}mv_a^2 = mgh_b + \frac{1}{2}mv_b^2 \qquad (1)$$

$$\text{or} \quad \Delta E_k + \Delta E_p = 0 \qquad (2)$$

In the roller coaster case, the total energy must be greater than mgd, where d (metres) is the diameter of the loop and zero potential energy is assigned to the lowest point, otherwise the carriage will be short of energy to reach the top of the loop.

Example 3 Due to excessive speed a car failed to make a turn and crashed through the railings at 72 km h^{-1} (20 ms^{-1}) on top of a 10 m cliff.
(a) At what speed did it hit the water?
(b) Would it make any difference whether the road inclined upwards or downwards at the crash site?

(a) Take 0 J as the gravitational potential energy at the surface of water.

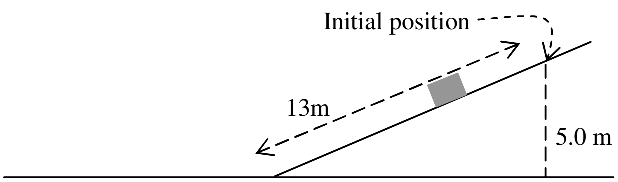
Use $mgh_a + \frac{1}{2}mv_a^2 = mgh_b + \frac{1}{2}mv_b^2$.

$0 + \frac{1}{2}mv_a^{\ 2} = m(9.8)(10) + \frac{1}{2}m(20^2)$,

$v_a = 24$ ms^{-1} was the speed of the car when it hit the water.

(b) The relationship $mgh_a + \frac{1}{2}mv_a^2 = mgh_b + \frac{1}{2}mv_b^2$ does not depend on the inclination of the road, $\therefore$ same speed.

Example 4 Due to handbrake-failure a parked 1000-kg mini-van rolled down an incline onto a horizontal straight stretch of road. It came to a stop after travelling 37 m along the horizontal section. See the diagram below for further information.
(a) Determine the average resistive force opposing its motion.
(b) Assume the same average resistive force on the slope. Find the speed of the van when it entered the horizontal section.



(a) When the van came to a stop the amount of heat generated by the resistive force (air resistance and rolling frictions) $= F_{resis,av} \times d$, where d is the total distance travelled.
At the initial position the total energy $= mgh$.
$\therefore F_{resis,av} \times d = mgh$, $F_{resis,av} \times (13+37) = 1000(9.8)(5.0)$,
$F_{resis,av} = 980$ N.

(b) $mgh_a + \frac{1}{2}mv_a^2 + heat = mgh_b + \frac{1}{2}mv_b^2$

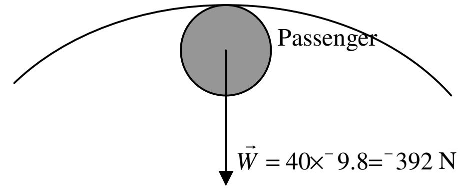
$0 + \frac{1}{2}(1000){v_a}^2 + 980\times 13 = 1000(9.8)(5.0) + 0$, $v_a \approx 8.5$ ms^{-1}.

Example 5 A carriage moves under gravity at 8.5 ms^{-1} at the top of a circular roller coaster loop of radius 7.5 m.
(a) Calculate the speed of the carriage at the bottom of the loop if there is no resistive force.
(b) In another run the total mass of the carriage with more passengers in it is doubled. The speed at the top remains the same. What is its speed at the bottom?
(c) Calculate the reaction force on a 40-kg passenger at the top.
(d) Calculate the reaction force on this passenger at the bottom if there is a resistive force restricting the speed to 11 ms^{-1} at the bottom.

(a) $mgh_a + \frac{1}{2}mv_a^2 = mgh_b + \frac{1}{2}mv_b^2$, $\therefore 2gh_a + v_a^2 = 2gh_b + v_b^2$.
$0 + v_a^2 = 2(9.8)(2\times 7.5) + 8.5^2$, $v_a \approx 19$ ms^{-1}.

(b) $2gh_a + v_a^2 = 2gh_b + v_b^2$ does not depend on the mass of the carriage if there is no resistive force, $\therefore$ same speed.

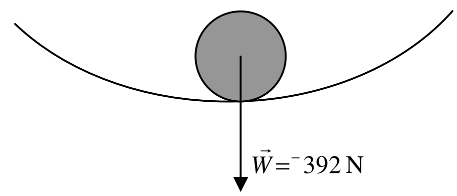
(c) Take upward as the positive direction. Let $\vec{R}$ (direction is unknown) be the reaction force on the passenger.



$\vec{F}_{net} = m\vec{a}$, $\vec{R} + {}^{-}392 = 40\times {}^{-}\left(\frac{8.5^2}{7.5}\right)$, $\vec{R} \approx {}^{+}7$ N.

The small upward reaction force is from the harness restraining the passenger from falling out of the seat.

(d)



$\vec{F}_{net} = m\vec{a}$, $\vec{R} + {}^{-}392 = 40\times {}^{+}\left(\frac{11^2}{7.5}\right)$, $\vec{R} \approx {}^{+}1000$ N.

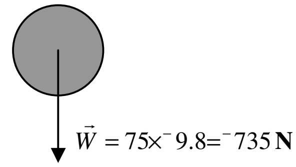
The large upward reaction force is from the seat providing the necessary force to keep the passenger in circular motion.

Example 6 During an air show a jet fighter plane makes a vertical circular manoeuvre. At the highest point of the loop (radius 1000 m) the plane flies at 99 ms^{-1} and the 75-kg pilot is in the upright position.
(a) Calculate the centripetal acceleration of the plane.
(b) What is the centripetal acceleration of the pilot?
(c) Determine the normal reaction on the pilot.

(a) $a = \frac{v^2}{r} = \frac{99^2}{1000} = 9.8$ ms^{-2}.

(b) 9.8 ms^{-2}.

(c) Let $\vec{R}$ (direction is unknown) be the reaction force on the passenger.



$\vec{F}_{net} = m\vec{a}$, $\vec{R} + {}^{-}735 = 75\times {}^{-}9.8$, $R \approx 0$ N.

The pilot feels weightless at the top of the loop.

Gravity

Newton's law of universal gravitation

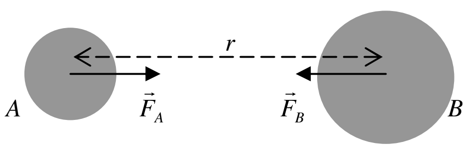
For any two objects, A and B, there is always a mutual attractive force *F* between them. *F* has the following properties.

$$F \propto M_A$$
$$F \propto M_B$$
$$F \propto \frac{1}{r^2}$$
$$\therefore F \propto \frac{M_A M_B}{r^2}$$
$$\therefore F = \frac{GM_A M_B}{r^2}$$

M_A and M_B are the masses of *A* and *B* in kg respectively, *r* metres is the separation between the centres of mass of the objects and, *G* is the constant of proportionality known as the **Universal constant**.

$$G = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$$

The equation $F = \frac{GM_A M_B}{r^2}$ is called **Newton's law of universal gravitation**.



The force of attraction is mutual, i.e. if *B* attracts *A* with force $\vec{F}_A$ then *A* attracts *B* with force $\vec{F}_B$, and $\vec{F}_A = -\vec{F}_B$ in accordance with Newton's third law and

$\left|\vec{F}_A\right| = \left|\vec{F}_B\right| = F = \frac{GM_A M_B}{r^2}$, according to Newton's law of universal gravitation.

Example 1 What is the average gravitational force between the earth (5.98×10^{24} kg) and the sun (1.99×10^{30} kg) with an average distance of 1.50×10^{11} m between them?

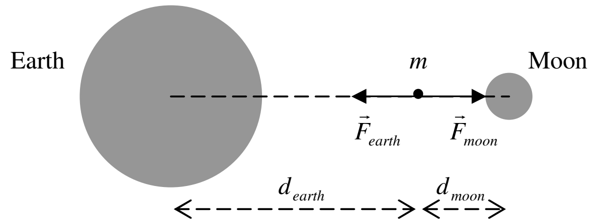
$$F = \frac{GM_A M_B}{r^2} = \frac{(6.67\times10^{-11})(5.98\times10^{24})(1.99\times10^{30})}{(1.50\times10^{11})^2}$$

$= 3.53\times10^{22}$ N

Example 2 There is a point between the earth and the moon (7.36×10^{22} kg) where a spacecraft experiences zero net gravitational force due to these two bodies only. The average distance between the earth and the moon is 3.82×10^{8} m.

(a) Find the value of the ratio $\frac{d_{earth}}{d_{moon}}$ at that point.

(b) How far is that point from the earth?



(a) $F_{moon} = F_{earth}$, $\frac{GM_{moon}m}{(d_{moon})^2} = \frac{GM_{earth}m}{(d_{earth})^2}$,

$$\frac{d_{earth}}{d_{moon}} = \sqrt{\frac{M_{earth}}{M_{moon}}} = \sqrt{\frac{5.98\times10^{24}}{7.36\times10^{22}}} \approx 9.$$

(b) $d_{earth} : d_{moon} \approx 9:1$,

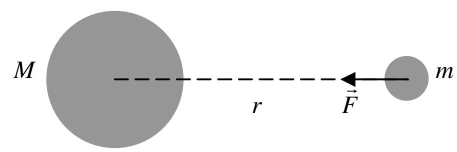
$\therefore d_{earth} \approx \frac{9}{9+1}\times3.82\times10^{8} \approx 3.4\times10^{8}$ m.

Gravitational field of a planet

It is defined as the gravitational force on each unit mass of an object *m* at some distance *r* from the planet *M*, i.e.

$$\vec{g} = \frac{\vec{F}}{m}.$$

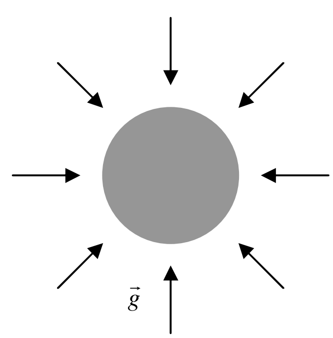
$\vec{g}$ is measured in N kg^{-1} when $\vec{F}$ and *m* are in N and kg respectively.



Since $F = \frac{GMm}{r^2}$, $\therefore g = \frac{F}{m} = \frac{GM}{r^2}$.

$\vec{g}$ is in the same direction as $\vec{F}$, i.e. towards the centre of mass of the planet.

Note: *M* in the above equation is the mass of the planet. $\vec{g}$ of the planet is due to the mass of the planet and it is independent of the object *m* in the vicinity of the planet.



When an object m is placed in the gravitational field $\vec{g}$ of a planet, the force of gravity on it is $m\vec{g}$ which is also known as the weight $\vec{W}$ of the object.

$$\vec{W} = m\vec{g}$$

Weight is a force and measured in newtons N.
$\vec{W}$ and $\vec{g}$ are in the same direction, i.e. towards the centre of the planet.

Example 3 Determine the gravitational field of the moon and the weight of a 5-kg object at its surface, using only the values of G, the mass and the radius of the moon.

$$g = \frac{GM_{moon}}{r^2} = \frac{(6.67\times10^{-11})(7.36\times10^{22})}{(1.74\times10^{6})^2} \approx 1.62 \text{ N kg}^{-1}.$$
$$W = mg = 5\times1.62 \approx 8.1\text{ N}.$$

Free fall and acceleration due to gravity

When an object moves in a gravitational field $\vec{g}$ and gravity is the **only** force acting on it, its acceleration is, according to Newton's second law,

$$\vec{a} = \frac{\vec{F}_{net}}{m} = \frac{m\vec{g}}{m} = \vec{g} \ .$$

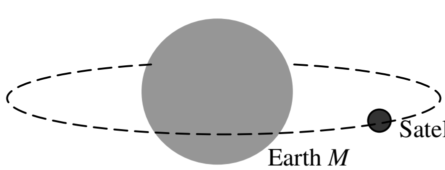
Acceleration $\vec{a}$ is in the same direction as $\vec{g}$.

Motion under the influence of gravity **only** is called **free fall** (not necessarily falling) and the acceleration is $\vec{g}$, e.g. orbiting satellites are in free fall.

A firing rocket in flight is not in free fall because gravity is not the only force on the rocket.

A sky diver is not in free fall because there is air resistance besides gravity.

Circular orbits under gravity



For a satellite in orbit around the earth, it is in free fall in the earth's gravitational field $\vec{g}$, its acceleration is $\vec{a} = \vec{g}$, where $g = \frac{GM}{r^2}$.

The satellite is in uniform circular motion. Therefore, acceleration is centripetal and $a = \frac{v^2}{r}$.

$$\therefore \frac{v^2}{r} = \frac{GM}{r^2}$$
$$\therefore v^2 r = GM$$

The last equation suggests that $v^2 r$ is a constant for any satellite around the earth. The constant is GM where M is the mass of the earth. Hence a satellite at a higher altitude has a lower speed.

For any two satellites, A and B, orbiting around the earth,

$$v_A^2 r_A = v_B^2 r_B \ .$$

Kepler's third law

Centripetal acceleration of a satellite can also be $a = \frac{4\pi^2 r}{T^2}$, where T is the period of revolution.

$$\therefore \frac{4\pi^2 r}{T^2} = \frac{GM}{r^2}$$
$$\therefore \frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

The last equation is known as **Kepler's third law**, which shows the relationship between r and T. $\frac{r^3}{T^2}$ is a constant which depends on the mass M of the planet supplying the gravitational field. At a higher altitude the period of the satellite is longer.

For any two satellites, A and B, orbiting around the same planet,

$$\frac{r_A^3}{T_A^2} = \frac{r_B^3}{T_B^2} \ .$$

Geostationary satellites

A geostationary satellite always appears to be at the same spot above the surface of the earth to an earth bound observer. It is directly above the equator and has the same period of revolution as the rotation of the earth, i.e. a day. There is one and **only one** orbit for all geostationary satellites.

Example 4 Calculate the radius of the orbit of a geostationary satellite using the known values of G and the mass of the earth.

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2},$$
$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\times10^{-11})(5.98\times10^{24})(24\times60\times60)^2}{4\pi^2}}$$
$$\approx 4.23\times10^{7}\text{ m}.$$

Planets in the solar system

The orbits of the planets in the solar system are elliptical, but can be approximated as circular with average radii to simplify calculations.

Example 5 The earth has an average orbital radius around the sun of 1.50×10^{11} m and its period of revolution is of course a year. Mars has an average orbital radius of 2.28×10^{11} m.
(a) Determine the period of Mars in earth years using only the given data for the earth and Mars.
(b) Hence calculate the magnitude of Mars' acceleration.
(c) What is the gravitational field strength in N kg^{-1} of the sun at where Mars is?

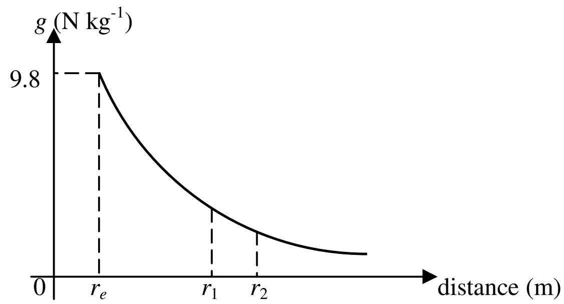
(a) $\frac{r_A^3}{T_A^2} = \frac{r_B^3}{T_B^2}$, $\left(\frac{T_M}{T_e}\right)^2 = \left(\frac{r_M}{r_e}\right)^3$, $\frac{T_M}{T_e} = \left(\frac{r_M}{r_e}\right)^{\frac{3}{2}}$,

$\therefore T_M = \left(\frac{r_M}{r_e}\right)^{\frac{3}{2}} T_e = \left(\frac{2.28\times10^{11}}{1.50\times10^{11}}\right)^{\frac{3}{2}} \times 1 = 1.87$ earth years.

(b) $a = \frac{4\pi^2 r_M}{{T_M}^2} = \frac{4\pi^2\left(2.28\times10^{11}\right)}{\left(1.87\times365\times24\times60\times60\right)^2} \approx 0.00258$ ms^{-2}.

(c) $g = a \approx 0.00258$ N kg^{-1}.

Gravitational field-distance graph (e.g. for the earth)



The distance is measured from the centre of the earth and r_e is the radius of the earth.

When an object moves from r_1 to r_2 under gravity only, its gravitational potential energy increases while its kinetic energy decreases by the same amount. The change in energy for *each kilogram* of the object is given by the area under the graph from r_1 to r_2. The area can be estimated from an accurately drawn graph.

Change in energy = mass of object × area under graph

$\Delta E = m \times$ area under g-x graph

If it moves from r_2 to r_1, potential energy decreases and kinetic energy increases.

To carry a satellite from the surface of the earth to an orbit of radius r_1, work (energy supplied to the satellite) is required and it is the area from r_e to r_1 for each kilogram of the satellite. Additional energy (kinetic) is required to make the satellite m to orbit around the earth M.

Kinetic energy $E_k = \frac{1}{2}mv^2$ and $v^2 r = GM$,

$\therefore E_k = \frac{1}{2}m\left(\frac{GM}{r}\right) = \frac{GMm}{2r}$ is the kinetic energy of an orbiting satellite.

Example 6 Given the values of G and the mass of the earth, calculate the kinetic energy of an earth satellite in circular orbit at an *altitude* of 3.60×10^4 km. The mass of the satellite is 350 kg and the radius of the earth is 6380 km.

Orbital radius $r = 6380 + 3.60\times10^4 = 4.238\times10^4$ km $= 4.238\times10^7$ m

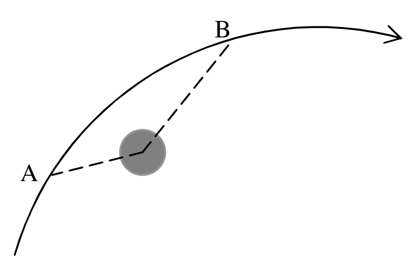
$\therefore E_k = \frac{GMm}{2r} = \frac{\left(6.67\times10^{-11}\right)\left(5.98\times10^{24}\right)(350)}{2\left(4.238\times10^7\right)} = 1.65\times10^9$ J

Example 7 Estimate the total amount of energy required to bring the satellite in example 6 from the surface of the earth to orbit around the planet.

From a suitable g-x graph, estimate the area under the graph from $x = 6.38\times10^6$ m to $x = 4.238\times10^7$ m, and multiply by 350 kg. This gives the energy required to take the 350-kg satellite to the specified altitude.

Estimated area $\approx 5.3\times10^7$ J kg^{-1}.
Energy required to take the 350-kg satellite to the specified altitude $\approx \left(5.3\times10^7\right)\times350 \approx 1.9\times10^{10}$ J.
Total energy required to place the satellite in orbit around the earth $\approx 1.9\times10^{10} + 1.65\times10^9 \approx 2\times10^{10}$ J.

Example 8 A piece of 250-kg space junk approaches the earth along an elliptical path from A to B. At A it has a speed of 4.0×10^4 ms^{-1}. At B its gravitational potential energy *increases* by 8.0×10^{10} J. Find its speed at B.



At A, $E_k = \frac{1}{2}(250)\left(4.0\times10^4\right)^2 = 2.0\times10^{11}$ J.

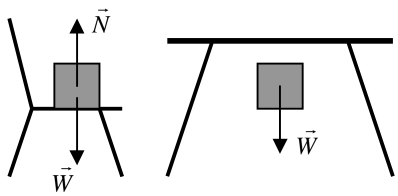
At B, $E_k = 2.0\times10^{11} - 8.0\times10^{10} = 1.2\times10^{11}$ J.

$\therefore \frac{1}{2}(250)v^2 = 1.2\times10^{11}$, $v \approx 3.1\times10^4$ ms^{-1}.

Weight and apparent weight

The weight $\vec{W}$ (measured in newtons) of an object of mass m (measured in kilograms) is the force of gravity on it and given by $\vec{W} = m\vec{g}$.

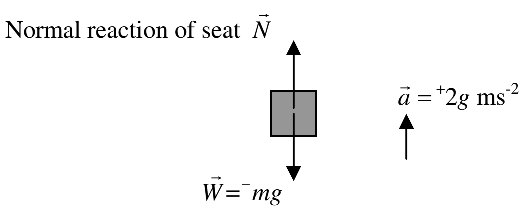
When you sit on a chair, you can feel your weight because the chair exerts a normal reaction force on you. However, when you jump off a table you are in free fall, and there is no reaction force on you and you feel weightless.



The 'weight' that you feel is called your **apparent weight**. It is measured by the normal reaction force $\vec{N}$. When you are at rest on a chair, your apparent weight is the same as your weight. When you are in free fall, your apparent weight is zero.

An astronaut of mass m kg in a space shuttle experiences an apparent weight which is different from the actual weight at takeoff.

Example 9 At takeoff the acceleration of the rocket carrying the space shuttle is $2g$ upward (taken as the positive direction). Find the apparent weight of the m-kg astronaut.



Apply Newton's second law to the astronaut:

$$\vec{F}_{net} = m\vec{a}$$
$$\vec{N} + \vec{W} = m\vec{a}$$
$$\vec{N} + {}^{-}mg = m\left({}^{+}2g\right)$$
$$\vec{N} = {}^{+}3mg$$

The apparent weight is three times the actual weight mg. Therefore the astronaut feels three times as heavy during takeoff as before takeoff.

Example 10 What is the apparent weight of the astronaut while the space shuttle is in a circular orbit around the earth?

While the space shuttle is in orbit, it is in free fall and has an acceleration of $\vec{g}$. Inside the space shuttle the astronaut has the same acceleration as the space shuttle. Therefore, the astronaut is in free fall and experiences weightlessness, i.e. the apparent weight of the astronaut is zero.