2008 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
C	E	D	A	C	A	C	A	A	B	E

12	13	14	15	16	17	18	19	20	21	22
C	A	C	A	A	D	A	B	E	B	C

Q1 The maximum value of $\left(\sin^{-1}x\right)^2-(\sin x)^{-2}$ is $\left(\sin^{-1}(1)\right)^2-(\sin(1))^{-2}\approx 1.316$. C

Q2 $\cos(x-y+z)=\cos(x-(y-z))$
$=\cos(x)\cos(y-z)+\sin(x)\sin(y-z)$. E

Q3 Equation of the inverse is $x=2\arctan\left[\frac{1}{2}(y+1)\right]-1$.
$\frac{x+1}{2}=\arctan\left[\frac{1}{2}(y+1)\right]$, $\frac{1}{2}(y+1)=\tan\left(\frac{x+1}{2}\right)$,
$y=2\tan\left(\frac{x+1}{2}\right)-1$, where $-\frac{\pi}{2}<\frac{x+1}{2}<\frac{\pi}{2}$, i.e.
$-\pi-1<x<\pi-1$. D

Q4 $y=\left(x-a+\frac{1}{\sqrt{x-a}}\right)\left(x-a-\frac{1}{\sqrt{x-a}}\right)+b$
$y=(x-a)^2-\frac{1}{x-a}+b$. Only one vertical asymptote $x=a$. A

Q5
$\frac{(x+1)^2}{4}+(y-2)^2=1\rightarrow\frac{x^2}{4}+(y-2)^2=1\rightarrow\frac{x^2}{4}+(y-1)^2=1$
$\rightarrow\frac{x^2}{4}+\left(\frac{y}{2}-1\right)^2=1$, i.e. $x^2+(y-2)^2=4$. C

Q6 $z=i\left(\cos\left(\frac{\pi}{2}+\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right)=i(-\sin\theta-i\cos\theta)$
$=\cos\theta-i\sin\theta=\cos(-\theta)+i\sin(-\theta)$. $\therefore Arg(z)=-\theta$. A

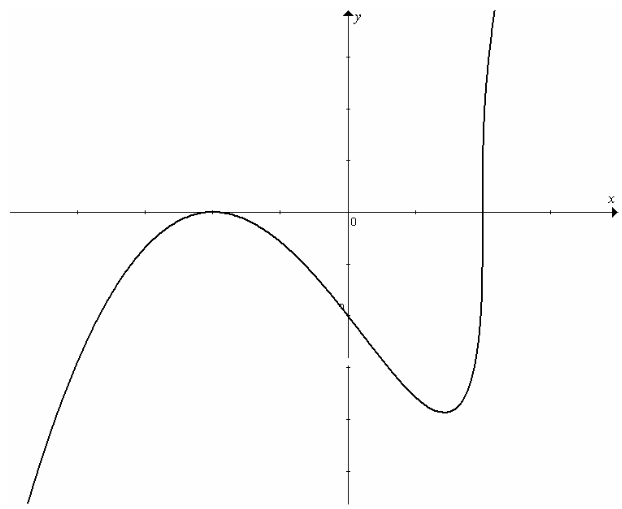
Q7 $2z^2-(5-i)z+3+11i=2(z-(3-2i))(z-(a+bi))$
$=2z^2-((2a+6)+(2b-4)i)z+(6a+4b)+(6b-4a)i$.
$\therefore 2a+6=5$ and $2b-4=-1$, $\therefore a=-\frac{1}{2}$ and $b=\frac{3}{2}$.

The other root is $-\frac{1}{2}+\frac{3}{2}i$. C

Q8 $3cis\left(\frac{\pi}{3}\right)-icis\left(-\frac{\pi}{6}\right)$
$=3\cos\left(\frac{\pi}{3}\right)+i3\sin\left(\frac{\pi}{3}\right)-i\cos\left(-\frac{\pi}{6}\right)+\sin\left(-\frac{\pi}{6}\right)$
$=\frac{3}{2}+\frac{3\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i+-\frac{1}{2}=1+i\sqrt{3}=2cis\left(\frac{\pi}{3}\right)$. A

Q9 A

Q10 Choose appropriate values of $a,b\in R$, use graphics calculator to draw the graph of $y=1000(x+a)^2(x-b)^{\frac{1}{3}}$. B



Q11 $\frac{d}{dx}\left(\frac{-1}{\sqrt{a-x^2}}\right)=\frac{d}{dx}\left(-\left(a-x^2\right)^{-\frac{1}{2}}\right)$
$=\frac{1}{2}\left(a-x^2\right)^{-\frac{3}{2}}(-2x)=-\frac{x}{\left(a-x^2\right)\sqrt{a-x^2}}$. E

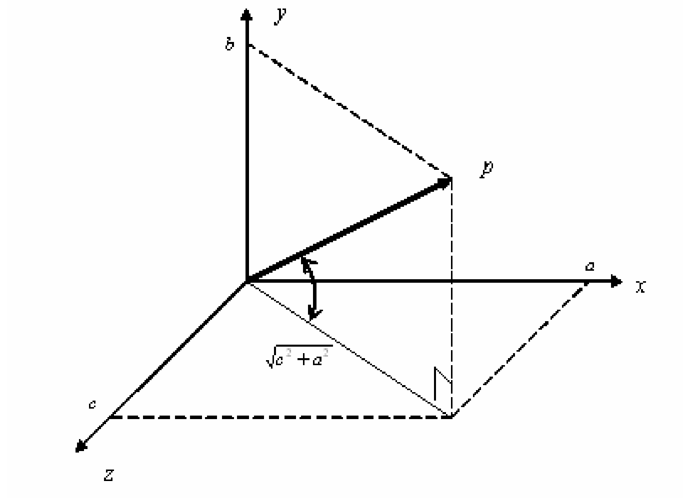
Q12 $2x^2y^2=x^2-y^2$, $4xy^2+2x^2 2y\frac{dy}{dx}=2x-2y\frac{dy}{dx}$,
$\therefore\frac{dy}{dx}=\frac{2x-4xy^2}{2y+4x^2y}=\frac{x-2xy^2}{y+2x^2y}$.
At $x=1$, $2(1)^2y^2=(1)^2-y^2$, $3y^2=1$, $y=\pm\frac{1}{\sqrt{3}}$,
and $\frac{dy}{dx}=\frac{1-\frac{2}{3}}{\pm\sqrt{3}}=\pm\frac{1}{3\sqrt{3}}$. C

Q13 Since $x\in R\setminus[-1,1]$, $\therefore x^2>1$,
$\therefore\int\frac{1}{1-x^2}d\left(x^2\right)=-\log_e\left|1-x^2\right|+c=-\log_e\left(x^2-1\right)+c$. A

Q14 $y'=y(a-y)$, when $y=2$,
gradient of slope (from graph) $=y'\approx\frac{2.0}{0.5}=4$,
$\therefore 4\approx 2(a-2)$, $\therefore a\approx 4$. C

Q15 $y=\int_{-\frac{1}{2}}^{-1}\tan\left(x^2\right)dx-1$. Use graphics calculator to evaluate the definite integral, $y\approx -0.356-1=-1.356$. A

Q16 A



Q17 $\boldsymbol{c}-3\boldsymbol{a}+2\boldsymbol{b}=\boldsymbol{0}$, $\therefore \boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are dependent. D

Q18 The direction of motion is given by the unit vector in the direction of the velocity vector.

$\boldsymbol{r}(t)=\tan(t)\boldsymbol{i}+\frac{1}{2}\sec^2(t)\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{r}(t)=\tan(t)\boldsymbol{i}+\frac{1}{2}\left(1+\tan^2(t)\right)\boldsymbol{j}+\boldsymbol{k}$

$\boldsymbol{u}(t)=\frac{d}{dt}\boldsymbol{r}(t)=\sec^2(t)\boldsymbol{i}+\tan(t)\sec^2(t)\boldsymbol{j}$

$=\left(1+\tan^2(t)\right)\boldsymbol{i}+\tan(t)\left(1+\tan^2(t)\right)\boldsymbol{j}$.

When $t=\frac{3\pi}{4}$, $\boldsymbol{u}=2\boldsymbol{i}-2\boldsymbol{j}$.

$\therefore\hat{\boldsymbol{u}}=\frac{1}{2\sqrt{2}}(2\boldsymbol{i}-2\boldsymbol{j})=\frac{\sqrt{2}}{2}\boldsymbol{i}-\frac{\sqrt{2}}{2}\boldsymbol{j}$. A

Q19 Initial velocity $=\frac{1}{0.20}(3\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$,

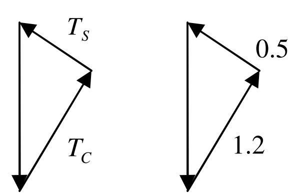
$\therefore$ initial speed $=\frac{1}{0.20}\sqrt{3^2+(-1)^2+1^2}=5\sqrt{11}$.

Final velocity $=\frac{1}{0.20}(\boldsymbol{i}-\boldsymbol{k})$,

$\therefore$ final speed $=\frac{1}{0.20}\sqrt{1^2+(-1)^2}=5\sqrt{2}$.

$\therefore$ change of speed $=5\sqrt{2}-5\sqrt{11}\approx -9.512$. B

Q20 $\sqrt{1.2^2+0.5^2}=1.3$, right-angled triangle.



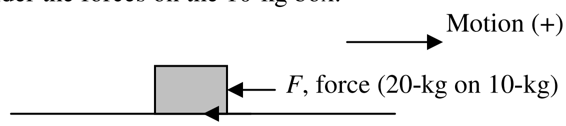
$\frac{T_S}{T_C}=\frac{0.5}{1.2}\approx 0.42$. E

Q21 The particle moves at constant speed and direction, $\therefore$ the resultant force on the particle is zero. Hence the reaction force of the slope on the particle is equal but opposite to the weight force on the particle. B

Q22 Consider the two boxes as a single object. Total force of friction $=30\times 0.2=6$ N opposite to motion,

$\therefore a=\frac{6}{30}=0.2$ ms^{-1} opposite to motion.

Consider the forces on the 10-kg box:



$R=ma$, ${}^-F+{}^-2=10\times{}^-0.2$, $\therefore F=0$. C

Section 2

Q1a. $a^2=1-v^2$, $a=\pm\sqrt{1-v^2}$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\pm\sqrt{1-v^2}$.

Q1bi. $\frac{1}{2}\frac{d\left(v^2\right)}{dx}=\pm\sqrt{1-v^2}$, $\frac{dx}{d\left(v^2\right)}=\pm\frac{1}{2\sqrt{1-v^2}}$,

$x=\mp\int\frac{1}{2\sqrt{1-\left(v^2\right)}}d\left(v^2\right)=\mp\int\frac{1}{2}\left(1-\left(v^2\right)\right)^{-\frac{1}{2}}d\left(v^2\right)=\mp\sqrt{1-v^2}+c$.

At $x=\pm1$, $v=0$, $\therefore c=0$, $\therefore x=\mp\sqrt{1-v^2}$.

Hence $v=\pm\sqrt{1-x^2}$. $\therefore A=\pm1$, $B=1$ and $C=-1$.

Q1bii. $v=\frac{dx}{dt}=\pm\sqrt{1-x^2}$, $\frac{dt}{dx}=\pm\frac{1}{\sqrt{1-x^2}}$. Note: both signs give the same result.

Choose $\frac{dt}{dx}=-\frac{1}{\sqrt{1-x^2}}$, $t=\int-\frac{1}{\sqrt{1-x^2}}dx=\cos^{-1}(x)+c$.

When $t=0$, $x=1$, $\therefore c=0$. $\therefore t=\cos^{-1}(x)$, where $0\le t\le\pi$.

Hence $x=\cos(t)$. $\therefore D=1$ and $n=1$.

Q1c. $\int_0^{\pi}v\,dt=\int_0^{\pi}\frac{dx}{dt}dt=\int_1^{-1}dx=\left[x\right]_1^{-1}=-2$.

Q1d. $x=\cos(t)$, $v=-\sin(t)$, $a=-\cos(t)$, $\therefore a=-x$.

Alternatively, refer to Q1a and b,

$a=\pm\sqrt{1-v^2}$ and $x=\mp\sqrt{1-v^2}$, $\therefore a=-x$.

Q2a. At $x=6$, $y=1+e$.

Area of shaded region $=\int_0^6 \left(1+e^{x-5}\right)dx+(1+e)\times 1$

$=\left[x+e^{x-5}\right]_0^6+1+e=6+e-e^{-5}+1+e=7+2e-e^{-5}$ m^2.

Q2bi $y=1+e^{x-5}$, $x=\log_e(y-1)+5$

At $x=0$, $y=1+e^{-5}$.

Max volume of water

$=\int_{1+e^{-5}}^{1+e}\pi x^2\,dy=\int_{1+e^{-5}}^{1+e}\pi(\log_e(y-1)+5)^2\,dy$ m^3.

Q2bii Volume of concrete

$=\pi 7^2(1+e)\ -\int_{1+e^{-5}}^{1+e}\pi(\log_e(y-1)+5)^2\,dy$

≈ 350 m^3. Evaluate the definite integral with graphics calculator.

Q2c. Max volume $=\int_{1+e^{-5}}^{1+e}\pi(\log_e(y-1)+5)^2\,dy=221.99$ m^3.

Time required $=\dfrac{221.99}{12}=18.499$ min.

Average rate of increase in depth of water

$=\dfrac{(1+e)-\left(1+e^{-5}\right)}{18.499}=0.146576$ m min^{-1} $=2.44$ mm s^{-1}.

Q2di. $V(h)=\int_{1+e^{-5}}^{h}\pi(\log_e(y-1)+5)^2\,dy$.

Q2dii. $\dfrac{dV}{dt}=\dfrac{dV}{dh}\times\dfrac{dh}{dt}$,

$\therefore \dfrac{dh}{dt}=\dfrac{\frac{dV}{dt}}{\frac{dV}{dh}}=\dfrac{12}{\dfrac{d}{dh}\left(\int_{1+e^{-5}}^{h}\pi(\log_e(y-1)+5)^2\,dy\right)}$.

Q3ai. $\boldsymbol{r}(t)=\left(t+1+\dfrac{1}{t+1}\right)\boldsymbol{i}+\left(t+1-\dfrac{1}{t+1}\right)\boldsymbol{j},\ t\ge 0$.

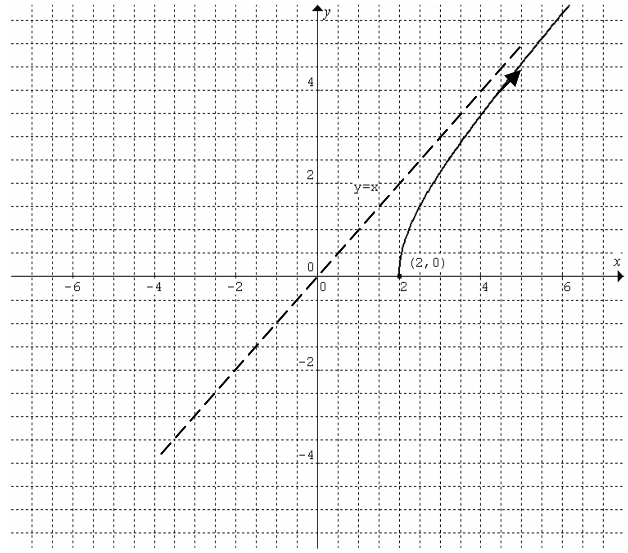
$x=t+1+\dfrac{1}{t+1}$, $y=t+1-\dfrac{1}{t+1}$, $\therefore x+y=2(t+1)$,

$\therefore t+1=\dfrac{x+y}{2}$, $\therefore x=\dfrac{x+y}{2}+\dfrac{2}{x+y}$, $x=\dfrac{(x+y)^2+4}{2(x+y)}$,

$\therefore 2x(x+y)=(x+y)^2+4$. Expand and simplify to $\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$.

Since $t\ge 0$, $\therefore x\ge 2$ and $y\ge 0$.

Q3b.



Q3ci. P$\left(\sqrt{2},3\right)$. Let $\left(t+1+\dfrac{1}{t+1},t+1-\dfrac{1}{t+1}\right)$ be the coordinates of the particle at time *t*.

Required vector $=\left(t+1+\dfrac{1}{t+1}-\sqrt{2}\right)\boldsymbol{i}+\left(t+1-\dfrac{1}{t+1}-3\right)\boldsymbol{j}$.

Q3cii. Distance $D=\sqrt{\left(t+1+\dfrac{1}{t+1}-\sqrt{2}\right)^2+\left(t+1-\dfrac{1}{t+1}-3\right)^2}$.

Use graphics calculator to find the shortest distance.

$D_{\min}=1.732$

Q3ciii. Time $=1.414$

Q3d. $\boldsymbol{v}(t)=\dfrac{d}{dt}\boldsymbol{r}(t)=\left(1-\dfrac{1}{(t+1)^2}\right)\boldsymbol{i}+\left(1+\dfrac{1}{(t+1)^2}\right)\boldsymbol{j},\ t\ge 0$.

At $t=\sqrt{2}$, $\boldsymbol{v}=\left(1-\dfrac{1}{\left(\sqrt{2}+1\right)^2}\right)\boldsymbol{i}+\left(1+\dfrac{1}{\left(\sqrt{2}+1\right)^2}\right)\boldsymbol{j}$.

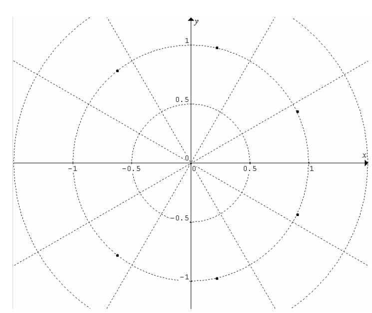
Speed $=\sqrt{0.828^2+1.172^2}\approx 1.4$.

Q4a. $\dfrac{z^7+1}{z+1}=\dfrac{(z+1)\left(z^6-z^5+z^4-z^3+z^2-z+1\right)}{z+1}$

$=z^6-z^5+z^4-z^3+z^2-z+1$.

$\therefore A=C=E=G=1$ and $B=D=F=-1$.

Q4b. Since $z^7+1=(z+1)P(z)$, the roots of $P(z)=0$ are the roots of $z^7+1=0$ except $z=-1$.



Q4c. $z = cis\left(\pm\frac{\pi}{7}\right), cis\left(\pm\frac{3\pi}{7}\right), cis\left(\pm\frac{5\pi}{7}\right)$.

Q4d. $P(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$
$= (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$,
$\therefore z_1 z_2 z_3 z_4 z_5 z_6 = 1$

Q4e. The z term in the expansion of $(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$ is $-(z_1 + z_2 + z_3 + z_4 + z_5 + z_6)z$. $\therefore z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 1$.

$$\therefore cis\left(-\frac{\pi}{7}\right) + cis\left(\frac{\pi}{7}\right) + cis\left(-\frac{3\pi}{7}\right) + cis\left(\frac{3\pi}{7}\right) + cis\left(-\frac{5\pi}{7}\right) + cis\left(\frac{5\pi}{7}\right) = 1$$

$$\therefore \cos\left(-\frac{\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right) + \cos\left(-\frac{3\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(-\frac{5\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right)$$

$$+ i\left(\sin\left(-\frac{\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) + \sin\left(-\frac{3\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) + \sin\left(-\frac{5\pi}{7}\right) + \sin\left(\frac{5\pi}{7}\right)\right) = 1$$

$$\therefore \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right)$$

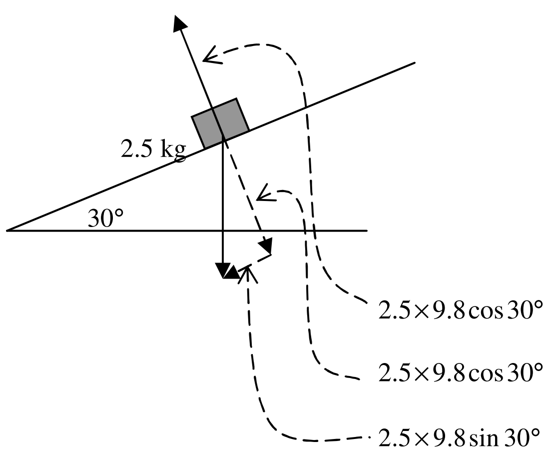
$$+ i\left(-\sin\left(\frac{\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) - \sin\left(\frac{3\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{5\pi}{7}\right) + \sin\left(\frac{5\pi}{7}\right)\right) = 1$$

$$\therefore 2\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{3\pi}{7}\right) + 2\cos\left(\frac{5\pi}{7}\right) = 1$$

$$\therefore \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}.$$

Q5a. The force of friction is zero.

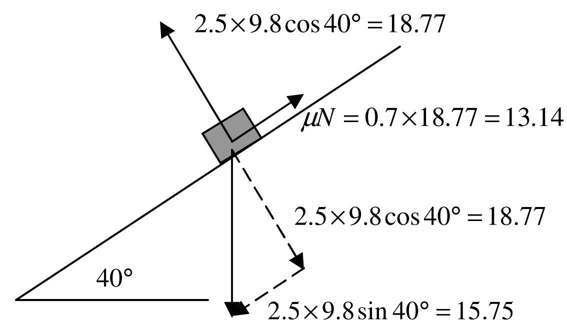
Q5b.



The component of the weight force along the plane is $2.5 \times 9.8 \sin 30° = 12.25$ N and it is less than the sliding friction $\mu N = 0.7 \times (2.5 \times 9.8 \cos 30°) = 14.85$ N.

$\therefore$ the object is at rest, and the friction preventing it from sliding is 12.25N.

Q5ci.

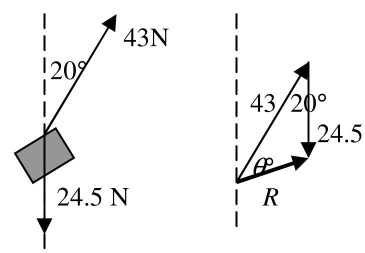


Resultant force $= {}^{+}15.75 + {}^{-}13.14 = {}^{+}2.61$ N,

$\therefore a = \frac{R}{m} = \frac{{}^{+}2.61}{2.5} = {}^{+}1.044 \approx {}^{+}1.04$ ms^{-2}.

Q5cii. $a = {}^{+}1.044$, $u = 0$, $s = {}^{+}3.0$, $v^2 = u^2 + 2as$,
$v = 2.503$ ms^{-1}.
$|\text{Momentum}| = mv = 2.5 \times 2.503 = 6.26$ kg ms^{-1}.

Q5d The 43 N pulling force has a vertical component of $43\cos 20° = 40.41$ N, and it is greater than the weight $2.5 \times 9.8 = 24.5$ N of the object. Hence the object takes off from the plank. The following diagram shows the two forces on the object.



$$R = \sqrt{43^2 + 24.5^2 - 2(43)(24.5)\cos 20°} = 21.664 \approx 21.7 \text{ N}$$

Q5e. $\frac{\sin\theta°}{24.5} = \frac{\sin 20°}{21.664}$, $\theta° = 22.8°$.
The direction of the resultant force R is $20° + 22.8° \approx 43°$ with the vertical, which is also the direction of motion.

Q5f. The object takes off from rest under a constant resultant force of 21.7 N at 43° with the vertical. It moves in a straight line into the air in the same direction as the resultant force and accelerates at $\frac{21.7}{25} \approx 8.7$ ms^{-2}.

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