VCAA Specialist Mathematics Exam 1 Solutions 2006

Q1a $2xy-9y^2+9=0$.

Implicit differentiation: $2y+2x\frac{dy}{dx}-18y\frac{dy}{dx}=0$,

$\therefore y+x\frac{dy}{dx}-9y\frac{dy}{dx}=0$, $(9y-x)\frac{dy}{dx}=y$,

$\therefore \frac{dy}{dx}=\frac{y}{9y-x}$.

Q1b When $y=1$, $x=0$, $\therefore \frac{dy}{dx}=\frac{1}{9}$.

Q2 $\frac{dy}{dx}=x\sqrt{x^2-16}$, $\therefore y=\int x\sqrt{x^2-16}\,dx$.

Let $u=x^2-16$, $\frac{du}{dx}=2x$, $x=\frac{1}{2}\frac{du}{dx}$.

$\therefore y=\int \frac{\sqrt{u}}{2}\frac{du}{dx}dx=\int \frac{\sqrt{u}}{2}du=\frac{u^{\frac{3}{2}}}{2\times\frac{3}{2}}+C=\frac{1}{3}\left(x^2-16\right)^{\frac{3}{2}}+C$.

Given $y=13$ when $x=5$, $\therefore C=4$.

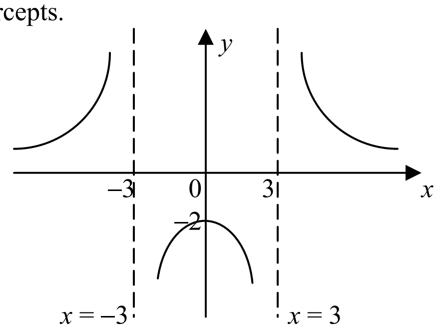
$\therefore y=\frac{1}{3}\left(x^2-16\right)^{\frac{3}{2}}+4$.

Q3a $y=\frac{36}{2x^2-18}=\frac{18}{x^2-9}=\frac{18}{(x-3)(x+3)}$

Asymptotes are $x=-3$ and $x=3$.

y-intercept: Let $x=0$, $\therefore y=-2$.

No x-intercepts.



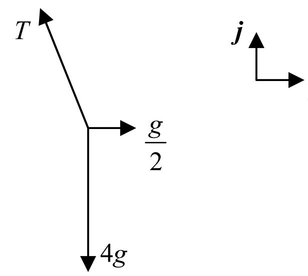
Q3b Partial fractions: $\frac{18}{(x-3)(x+3)}=\frac{3}{x-3}-\frac{3}{x+3}$.

$\text{Area}=-2\times\int_0^2\left(\frac{3}{x-3}-\frac{3}{x+3}\right)dx=-6\int_0^2\left(\frac{1}{x-3}-\frac{1}{x+3}\right)dx$

$=-6\left[\log_e|x-3|-\log_e|x+3|\right]_0^2=-6\left[\log_e\frac{|x-3|}{|x+3|}\right]_0^2$

$=-6\log_e\frac{|-1|}{|5|}=-6\log_e\left(\frac{1}{5}\right)=6\log_e 5$.

Q4a



Q4b $\boldsymbol{R}=\boldsymbol{0}$

$\boldsymbol{T}+\frac{g}{2}\boldsymbol{i}-4g\boldsymbol{j}=\boldsymbol{0}$, $\therefore \boldsymbol{T}=-\frac{g}{2}\boldsymbol{i}+4g\boldsymbol{j}$,

$\therefore T=\sqrt{\left(-\frac{g}{2}\right)^2+(4g)^2}=g\sqrt{\frac{1}{4}+16}=g\sqrt{\frac{65}{4}}=\frac{g\sqrt{65}}{2}$.

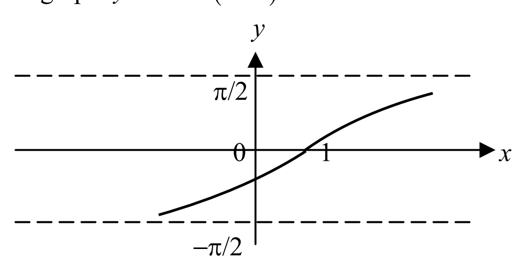
Q5a $\tan\left(\frac{\pi}{4}\right)=\tan\left(2\times\frac{\pi}{8}\right)=\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$,

$\therefore 1=\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$, $1-\tan^2\left(\frac{\pi}{8}\right)=2\tan\left(\frac{\pi}{8}\right)$,

$\therefore \tan^2\left(\frac{\pi}{8}\right)+2\tan\left(\frac{\pi}{8}\right)-1=0$, a quadratic equation.

$\therefore \tan\left(\frac{\pi}{8}\right)=\frac{-2+\sqrt{2^2-4(1)(-1)}}{2(1)}=\sqrt{2}-1$, since $\tan\left(\frac{\pi}{8}\right)>0$.

Q5b The graph $y=\tan^{-1}(x-1)$ is shown below.

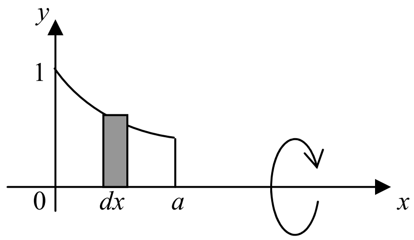


$\therefore \tan^{-1}(x-1)+a\tan\left(\frac{\pi}{8}\right)>0$ for all x if $a\tan\left(\frac{\pi}{8}\right)\ge\frac{\pi}{2}$,

i.e. $a\left(\sqrt{2}-1\right)\ge\frac{\pi}{2}$, $\therefore a\ge\frac{\pi}{2\left(\sqrt{2}-1\right)}$.

Hence minimum value of $a=\frac{\pi}{2\left(\sqrt{2}-1\right)}=\frac{\pi\left(\sqrt{2}+1\right)}{2}$.

Q6a



$$V=\int_0^a \pi y^2\,dx=\int_0^a \pi\left(e^{-x}\right)^2 dx=\int_0^a \pi e^{-2x}\,dx$$

Q6b $V=\left[\dfrac{\pi e^{-2x}}{-2}\right]_0^a=\left(\dfrac{\pi e^{-2a}}{-2}\right)-\left(\dfrac{\pi e^{0}}{-2}\right)=\dfrac{\pi}{2}\left(1-e^{-2a}\right)$

Q6c $\dfrac{\pi}{2}\left(1-e^{-2a}\right)=\dfrac{5\pi}{18}$, $1-e^{-2a}=\dfrac{5}{9}$, $e^{-2a}=\dfrac{4}{9}$, $e^{2a}=\dfrac{9}{4}$,

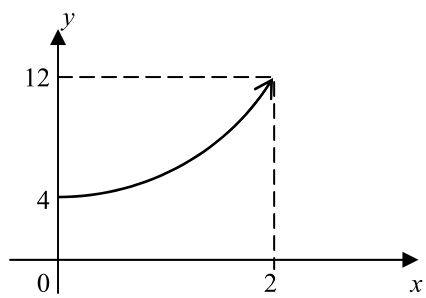
$2a=\log_e\left(\dfrac{9}{4}\right)$, $a=\dfrac{1}{2}\log_e\left(\dfrac{9}{4}\right)=\log_e\left(\dfrac{3}{2}\right)$.

Q7a $x=\sqrt{t-2}$, $y=2t$ for $2\le t\le 6$.

Eliminate t: $x^2=t-2$, $\therefore t=x^2+2$,

$\therefore y=2\left(x^2+2\right)$ for $0\le x\le 2$ and $4\le y\le 12$.

Q7b



Q8 $\displaystyle\int\frac{2+6x}{\sqrt{4-x^2}}dx=\int\frac{2}{\sqrt{4-x^2}}dx+\int\frac{6x}{\sqrt{4-x^2}}dx$

$=2\sin^{-1}\left(\dfrac{x}{2}\right)+\displaystyle\int\frac{-3}{\sqrt{u}}\frac{du}{dx}dx$ (where $u=4-x^2$ and $\dfrac{du}{dx}=-2x$)

$=2\sin^{-1}\left(\dfrac{x}{2}\right)-3\displaystyle\int u^{-\frac{1}{2}}du$

$=2\sin^{-1}\left(\dfrac{x}{2}\right)-6u^{\frac{1}{2}}$

$=2\sin^{-1}\left(\dfrac{x}{2}\right)-6\sqrt{4-x^2}$.

Q9a $1+i\sqrt{3}$ is in the first quadrant of the complex plane.

$r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$, $\theta=Tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)=\dfrac{\pi}{3}$,

$\therefore 1+i\sqrt{3}=2cis\left(\dfrac{\pi}{3}\right)$.

Q9b $z^2+2z-i\sqrt{3}=0$

Complete the square: $z^2+2z+1-1-i\sqrt{3}=0$,

$\therefore (z+1)^2-\left(1+i\sqrt{3}\right)=0$,

$(z+1)^2=1+i\sqrt{3}$.

In polar form: $(z+1)^2=2cis\left(\dfrac{\pi}{3}+2k\pi\right)$.

$\therefore z+1=\sqrt{2}cis\dfrac{1}{2}\left(\dfrac{\pi}{3}+2k\pi\right)$.

Let $k=0$, $z+1=\sqrt{2}cis\left(\dfrac{\pi}{6}\right)=\dfrac{\sqrt{6}}{2}+\dfrac{\sqrt{2}}{2}i$,

$\therefore z=\left(\dfrac{\sqrt{6}-2}{2}\right)+\dfrac{\sqrt{2}}{2}i$.

Let $k=-1$,

$z+1=\sqrt{2}cis\left(\dfrac{\pi}{6}-\pi\right)=\sqrt{2}cis\left(-\dfrac{5\pi}{6}\right)=-\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{2}}{2}i$,

$\therefore z=\left(\dfrac{-\sqrt{6}-2}{2}\right)-\dfrac{\sqrt{2}}{2}i$.

Alternative method: Let $z=a+bi$ where $a,b\in R$, be a solution to the equation $z^2+2z-i\sqrt{3}=0$.

$\therefore (a+bi)^2+2(a+bi)-i\sqrt{3}=0$.

Expand and collect real parts and imaginary parts:

$\left(a^2+2a-b^2\right)+\left(2ab+2b-\sqrt{3}\right)i=0$.

$\therefore a^2+2a-b^2=0$ (1)

and $2ab+2b-\sqrt{3}=0$ (2)

From (1), $a^2+2a=b^2$ (3)

From (2), $2b(a+1)=\sqrt{3}$, $\therefore 4b^2(a+1)^2=3$,

$\therefore 4b^2\left(a^2+2a+1\right)=3$ (4)

Substitute (3) in (4), $4b^2\left(b^2+1\right)=3$, $\therefore 4\left(b^2\right)^2+4\left(b^2\right)-3=0$,

$\therefore \left(2b^2+3\right)\left(2b^2-1\right)=0$.

Since $2b^2+3>0$, $\therefore 2b^2-1=0$, $\therefore b=\pm\dfrac{\sqrt{2}}{2}$.

$2b(a+1)=\sqrt{3}$, $\therefore a=\dfrac{\sqrt{3}}{2b}-1=\pm\sqrt{\dfrac{3}{2}}-1=\dfrac{\pm\sqrt{6}-2}{2}$.

Hence the solutions are:

$z=\left(\dfrac{\sqrt{6}-2}{2}\right)+\dfrac{\sqrt{2}}{2}i$ or $z=\left(\dfrac{-\sqrt{6}-2}{2}\right)-\dfrac{\sqrt{2}}{2}i$.

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