2008 Mathematical Methods 3/4 Trial Exam 2 Solutions
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SECTION 1

1	2	3	4	5	6	7	8	9	10	11
C	A	E	B	D	D	E	D	B	D	A

12	13	14	15	16	17	18	19	20	21	22
B	E	A	B	A	A	C	A	A	D	C

Q1 $f(x)=\tan\left(\frac{x}{2}\right)$ is defined over D. C

Q2 $b=2\log_2\left(\frac{a}{2}\right)$, $\frac{b}{2}=\log_2\left(\frac{a}{2}\right)$, $\frac{a}{2}=2^{\frac{b}{2}}$,

$a=2^{\frac{b}{2}}\times 2=2^{\frac{b}{2}+1}=2^{\frac{1}{2}(b+2)}=\left(e^{\log_e 2}\right)^{\frac{1}{2}(b+2)}=e^{\frac{1}{2}(b+2)\log_e 2}$. A

Q3 $e^{2x}-2e^x+k=0$, $\left(e^x\right)^2-2\left(e^x\right)+k=0$,

$e^x=\frac{2\pm\sqrt{4-4k}}{2}=1\pm\sqrt{1-k}$.

Two solutions exist when $1-\sqrt{1-k}>0$ and $1-k>0$.

$\therefore 1>\sqrt{1-k}$ and $k<1$, i.e. $1>1-k$ and $k<1$.

Hence $0<k<1$. E

Q4 $f(x)=2\cos(3x)$, $\frac{1}{4}f\left(\frac{\pi}{6}-\frac{1}{3}x\right)=\frac{1}{4}\left(2\cos\left(3\left(\frac{\pi}{6}-\frac{1}{3}x\right)\right)\right)$,

$=\frac{1}{2}\cos\left(\frac{\pi}{2}-x\right)=\frac{1}{2}\sin x$. B

Q5 The graph is part of a circle centred at $(0,0)$ and with a radius of 1 unit.

$x^2+y^2=1$, $y=\sqrt{1-x^2}$, $\therefore a=1$ and $b=1$. D

Q6 Transform e^x to $\left|a-be^{-x}\right|-c$ graphically, or choose positive a, b and c values, and use graphics calculator to sketch graph. D

Q7 $\left(3\sqrt{x}+x\right)\left(3\sqrt{x}-x\right)=9x-x^2$,

$\left(1-x\sqrt{2}\right)\left(2+x\sqrt{2}\right)=2-x\sqrt{2}-2x^2$,

$\sqrt[3]{x^3-3x^2+3x-1}=\sqrt[3]{(x-1)^3}=x-1$,

$\frac{x^{\frac{3}{2}}-(2x)^{\frac{5}{2}}}{x^{-\frac{3}{2}}}=x^3-2^{\frac{5}{2}}x^4$. E

Q8 D

Q9 The range of g is $(-1,0]$, $\therefore$ the range of $f\circ g$ is $(0,1]$. B

Q10 $f\left(\frac{x}{y}\right)=1-\sqrt{\frac{x}{y}}=1-\frac{\sqrt{x}}{\sqrt{y}}=\frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}}$.

$\frac{f(x)-f(y)}{1-f(y)}=\frac{1-\sqrt{x}-\left(1-\sqrt{y}\right)}{1-\left(1-\sqrt{y}\right)}=\frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}}$. D

Q11 A

Q12 $P(x)=\frac{f(\log_e x)}{g\left(\sqrt{x}\right)}$, use the quotient rule and the chain rule,

$$P'(x)=\frac{\left(g\left(\sqrt{x}\right)\right)\left(\frac{1}{x}\right)\left(f'(\log_e x)\right)-\left(f(\log_e x)\right)\left(\frac{1}{2\sqrt{x}}\right)\left(g'\left(\sqrt{x}\right)\right)}{\left[g\left(\sqrt{x}\right)\right]^2}$$

$=\frac{2\sqrt{x}\,g\left(\sqrt{x}\right)f'(\log_e x)-xf(\log_e x)g'\left(\sqrt{x}\right)}{2x\sqrt{x}\left[g\left(\sqrt{x}\right)\right]^2}$. B

Q13 $y=2x^3-3ax^2+5$, $\frac{dy}{dx}=6x^2-6ax=6x(x-a)$. $\therefore$ it is a stationary point at $x=a$. Equation of the normal at $x=a$ is $x=a$. E

Q14 Use graphics calculator, at $x=\frac{3}{4}$, $\frac{dy}{dx}=2.373$. A

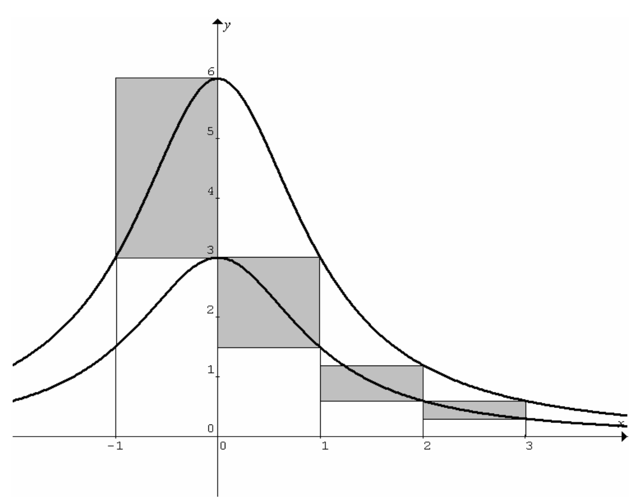
Q15 $f(x)=\sqrt{1-x}$, $f'(x)=\frac{-1}{2\sqrt{1-x}}$. Let $x=-3$ and $h=-0.1$.

$f(x+h)\approx f(x)+hf'(x)$

$\therefore f(-3.1)\approx f(-3)+(-0.1)f'(-3)$

$f(-3.1)\approx\sqrt{1-(-3)}+(-0.1)\left(\frac{-1}{2\sqrt{1-(-3)}}\right)=\frac{81}{40}$. B

Q16 $3\times1+1.6\times1+0.5\times1+0.3\times1\approx5.4$ A



Q17 $f(x)=(x+2)g(x)$ is a continuous decreasing function, $\therefore$ it has only one x-intercept at $x=-2$.

Required area $=\int_a^{-2} f(x)dx-\int_{-2}^{b} f(x)dx=[F(x)]_a^{-2}-[F(x)]_{-2}^{b}$

$=2F(-2)-F(a)-F(b)$. A

Q18 $\frac{20}{36}=\frac{5}{9}$. C

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Q19

$\overline{X}=(-1)\left(2\times\frac{1}{30}\right)+1\left(2\times\frac{1}{10}\right)+3\left(2\times\frac{1}{6}\right)+5\left(2\times\frac{2}{15}\right)+7\left(2\times\frac{1}{15}\right)$

$=\frac{51}{15}$. A

Q20

	$X>0.36$	$X<0.36$	
$X>0.64$	0.19	0	0.19
$X<0.64$	0.33	0.48	0.81
	0.52	0.48	1

$\Pr(X>0.36 \mid X<0.64)=\frac{\Pr(0.36<X<0.64)}{\Pr(X<0.64)}=\frac{0.33}{0.81}=\frac{11}{27}$. A

Q21 $\mu=\frac{1.25+1.35}{2}=1.30$ and $\sigma=\frac{1.35-1.30}{2}=0.025$.

$\Pr(1.28<X<1.38)=normalcdf(1.28,1.38,1.30,0.025)\approx 0.79$.

D

Q22 $\int_0^M \frac{1}{\sqrt{3}\cos^2 t}dt=0.5$, $\int_0^M \frac{1}{\sqrt{3}}\sec^2 t dt=0.5$,

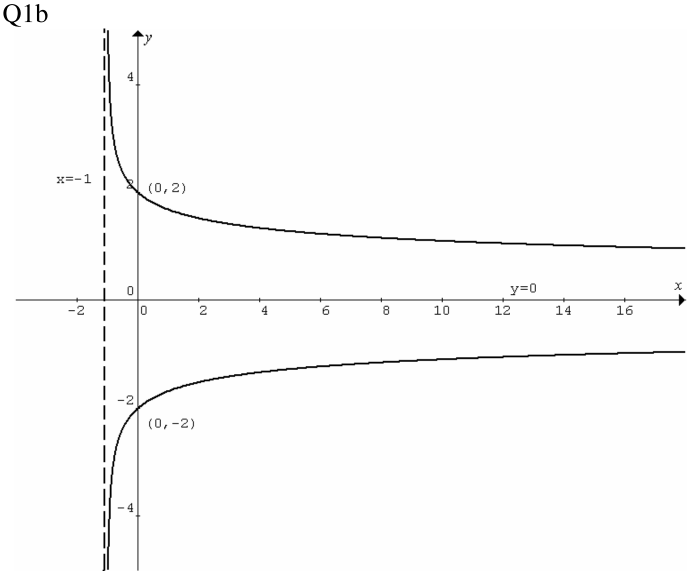
$\therefore \left[\frac{1}{\sqrt{3}}\tan t\right]_0^M=0.5$, $\frac{1}{\sqrt{3}}\tan M=0.5$, $M=0.714$. C

SECTION 2

Q1a $x=\frac{16}{y^4}-1$, $x+1=\frac{16}{y^4}$, $y^4=\frac{16}{x+1}$, $y=\pm\left(\frac{2^4}{x+1}\right)^{\frac{1}{4}}$,

$y=\pm\frac{2}{(x+1)^{\frac{1}{4}}}$.

Q1b



Q1c $L(x)=\frac{2}{(x+1)^{\frac{1}{4}}}-\frac{-2}{(x+1)^{\frac{1}{4}}}=\frac{4}{(x+1)^{\frac{1}{4}}}$.

Q1di $\frac{dL}{dt}=\frac{dL}{dx}\times\frac{dx}{dt}=-\frac{1}{4}\times 4(x+1)^{-\frac{5}{4}}\times 2=-\frac{2}{(x+1)^{\frac{5}{4}}}$.

Rate of decrease of $L=\frac{2}{(x+1)^{\frac{5}{4}}}$.

Q1dii When $t=7.5$, $x=vt=2\times 7.5=15$.

Rate of decrease of $L=\frac{2}{(15+1)^{\frac{5}{4}}}=\frac{1}{16}$.

Q1e $\Delta A=2\times\int_0^{15}\frac{2}{(x+1)^{\frac{1}{4}}}dx=2\times\left[\frac{8(x+1)^{\frac{3}{4}}}{3}\right]_0^{15}=\frac{112}{3}$.

Average rate of increase of the area $=\frac{\Delta A}{\Delta t}=\frac{\frac{112}{3}}{7.5}=\frac{224}{45}$.

Q2a $2\pi\left[1-(1-h)^2\right]=2\pi(1-(1-h))(1+(1-h))=2\pi h(2-h)$.

Q2b $\frac{2}{3}-(1-h)+\frac{(1-h)^3}{3}=\frac{2}{3}-1+h+\frac{1-3h+3h^2-h^3}{3}$

$=h^2-\frac{h^3}{3}$.

Q2c When $h=1$, max $V=2\pi\left(\frac{2}{3}\right)=\frac{4\pi}{3}$ m^3.

Q2d $V=\frac{4\pi}{3}$ m^3 $=\frac{4\pi}{3}\times 10^6$ cm^3 $=\frac{4\pi}{3}\times 10^3$ litres.

Time required $=\frac{\frac{4\pi}{3}\times 10^3}{2}=\frac{2\pi}{3}\times 10^3$ seconds.

Q2e 2 litres per second $=2\times 10^{-3}$ m^3 per second.

$V=2\pi\left(h^2-\frac{h^3}{3}\right)$, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$,

$\frac{dV}{dt}=2\pi\left(2h-h^2\right)\frac{dh}{dt}$, $\therefore \frac{dh}{dt}=\frac{1}{2\pi\left(2h-h^2\right)}\times\frac{dV}{dt}$.

When $h=0.5$, $\frac{dh}{dt}=\frac{1}{2\pi(1-0.25)}\times\left(-2\times 10^{-3}\right)=-1.82\times 10^{-4}$

Rate of decrease $=1.82\times 10^{-4}$ ms^{-1}.

Q2f $A=2\pi\left(2h-h^2\right)$, $\frac{dA}{dt}=\frac{dA}{dh}\times\frac{dh}{dt}=2\pi(2-2h)\frac{dh}{dt}$.

When $h=0.5$, $\frac{dA}{dt}=2\pi\times\left(-1.82\times 10^{-4}\right)=-1.14\times 10^{-3}$.

Rate of decrease $=1.14\times 10^{-3}$ m^{2}s^{-1}.

Q2gi When $h=0.5$,

volume of water $=2\pi\left(0.5^2-\frac{0.5^3}{3}\right)=1.309$ m^3.

Volume of water plus pebbles $=1.309+0.831=2.140$ m^3.

$\therefore 2.140=2\pi\left(h^2-\frac{h^3}{3}\right)$. Using graphics calculator, $h=0.661$ m.

Q2gii Yes. $\frac{dh}{dt}=\frac{dV}{dt}\times\frac{1}{2\pi\left(2h-h^2\right)}$.

Since $\frac{dV}{dt}$ is constant, $\frac{dh}{dt}\propto\frac{1}{2h-h^2}$.

Q3a For $y=c-a\cos(bx)$, $\frac{T}{2}=6$, $\therefore T=12=\frac{2\pi}{b}$, $\therefore b=\frac{\pi}{6}$.

Its amplitude is 2, $\therefore a=2$. $y=-2\cos\left(\frac{\pi}{6}x\right)$ is translated upwards by 3, $\therefore c=3$.

For semi-circle $(x-h)^2+(y-k)^2=1.5^2$, $x\in[6.5,8]$, radius is 1.5. It is the translation of the semi-circle $x^2+y^2=1.5^2$, $x\in[-1.5,0]$ to the right by 8 units and upwards by 2.5 units.

$\therefore h=8$ and $k=2.5$.

Q3b $P(8,4)$

Q3ci The semi-circle in the third wave crest is the translation of the semi-circle $x^2+y^2=1.5^2$, $x\in[-1.5,0]$ to the right by 24 units and upwards by 2.5 units,

$\therefore$its equation is $(x-24)^2+(y-2.5)^2=1.5^2$, $x\in[22.5,24]$.

Q3cii The cosine curve in the fourth wave crest is the translation of $y=3-2\cos\left(\frac{\pi}{6}x\right)$, $x\in[0,8]$, to the right by 24 units, $\therefore$its equation is $y=3-2\cos\left(\frac{\pi}{6}(x-24)\right)$, $x\in[24,32]$.

Q3di Same area as the first wave crest,

$A=\int_0^8\left(3-2\cos\left(\frac{\pi}{6}x\right)-(-2)\right)dx-\frac{1}{2}\pi(1.5)^2$

$=\int_0^8\left(5-2\cos\left(\frac{\pi}{6}x\right)\right)dx-\frac{9\pi}{8}=\left[5x-\frac{12\sin\left(\frac{\pi}{6}x\right)}{\pi}\right]_0^8-\frac{9\pi}{8}$

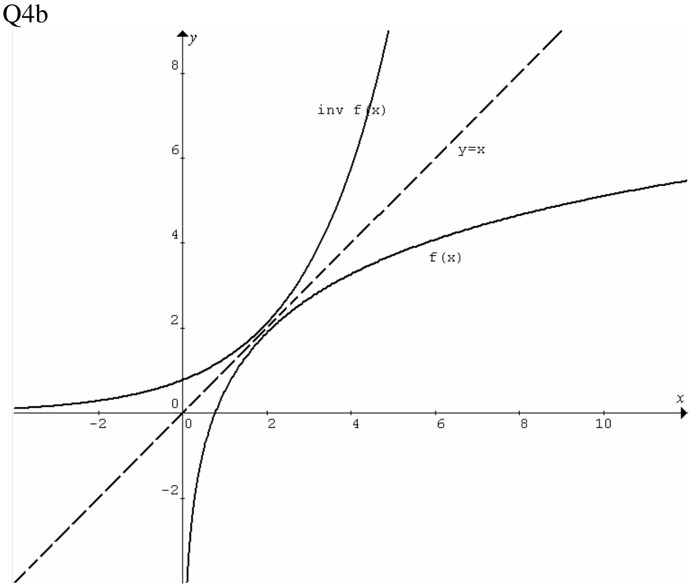
$=40+\frac{6\sqrt{3}}{\pi}-\frac{9\pi}{8}$ m^2.

Q3dii $V=A\times l=\left(40+\frac{6\sqrt{3}}{\pi}-\frac{9\pi}{8}\right)\times 10=398$ m^3.

Q4a Inverse equation: $x=a\log_e y+\frac{1}{2}$, $\log_e y=\frac{x-\frac{1}{2}}{a}$,

$y=e^{\frac{x-\frac{1}{2}}{a}}=e^{\frac{2x-1}{2a}}$, $\therefore f^{-1}(x)=e^{\frac{2x-1}{2a}}$.

Q4b



Q4ci $y=a\log_e x+\frac{1}{2}$. At the contact point, $\frac{dy}{dx}=\frac{a}{x}=1$ and $y=x$, $\therefore y=x=a$, $\therefore a=a\log_e a+\frac{1}{2}$, $\therefore a-a\log_e a-\frac{1}{2}=0$.

Using graphics calculator, $a=2.156$.

Q4cii $(2.156,2.156)$.

Q5a $1000 \times \Pr(X > 375) = 1000 \times \int_{375}^{\infty} e^{-\pi(x-375.5)^2}\,dx = 895$. Note: let ∞ be 380, use graphics calculator to evaluate the definite integral.

Q5b $1000 \times \Pr(X > 375.3) = 1000 \times \int_{375.3}^{\infty} e^{-\pi(x-375.5)^2}\,dx = 692$.

Translate the pdf and all x-values to the left by 0.3,

$\therefore 1000 \times \Pr(X > 375) = 1000 \times \int_{375}^{\infty} e^{-\pi(x-375.2)^2}\,dx = 692$.

$\therefore k = 375.2$.

Q5c Binomial, $n = 5$, $p = 1 - 0.692 = 0.308$,
$\Pr(X = 4) = binompdf(5,0.308,4) = 0.031$.

Q5d Since the machines are identical, it makes no difference to the probability which machine the cans are selected from.
$\Pr(X = 8) = binompdf(10,0.308,8) = 0.002$.

Q5e Given that 2 cans were under and 1 can was over 375 ml,
$\therefore$ the probability that 6 of the remaining 7 cans are under 375 ml
$= binompdf(7,0.308,6) = 0.004$.

Q5fi Symmetric bell shape.

Q5fii

If Y is normally distributed, then $\sigma_Y = \frac{376.85 - 375.5}{2} = 0.675$.

$\Pr(\mu - \sigma < Y < \mu + \sigma)$
$= \frac{normalcdf(374.825,376.175,375.5,0.3989) + normalcdf(374.825,376.175,375.5,0.7978)}{2}$
$= 0.756 \neq 0.683$.

$\Pr(\mu - 3\sigma < Y < \mu + 3\sigma)$
$= \frac{normalcdf(373.475,377.525,375.5,0.3989) + normalcdf(373.475,377.525,375.5,0.7978)}{2}$
$= 0.994 \approx 0.997$.

$\therefore Y$ is **not** exactly normally distributed.

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