2008 VCAA Specialist Mathematics Exam 1 Solutions
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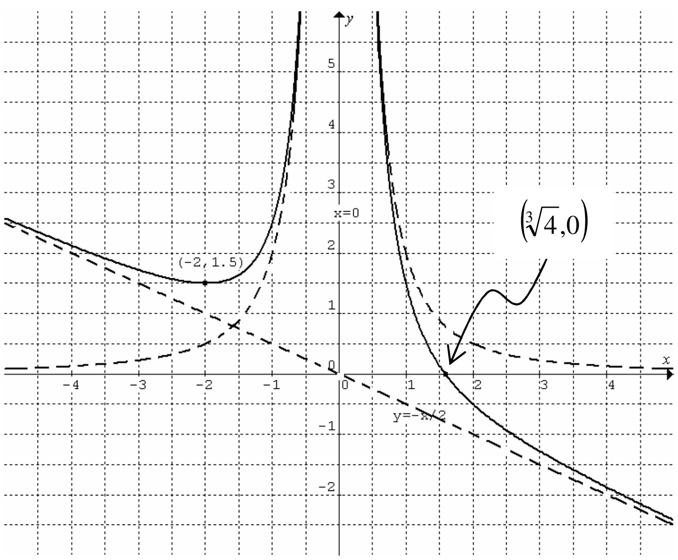
Q1 Sketch $y = \frac{2}{x^2}$ and $y = -\frac{x}{2}$, then use addition of ordinates to sketch $y = \frac{2}{x^2} - \frac{x}{2}$.

Straight line asymptotes are $x = 0$ and $y = -\frac{x}{2}$.

x-intercept: Let $y = 0$, $\frac{2}{x^2} - \frac{x}{2} = 0$, $4 - x^3 = 0$, $x = \sqrt[3]{4}$, $\left(\sqrt[3]{4}, 0\right)$

Turning point: $\frac{dy}{dx} = 0$, $-\frac{4}{x^3} - \frac{1}{2} = 0$, $x^3 = -8$, $x = -2$ and $y = \frac{3}{2}$, $(-2, 1.5)$.



Q2 When $x = 1$, $3 + 2y + y^2 = 11$, $y^2 + 2y - 8 = 0$, $(y + 4)(y - 2) = 0$, $\therefore y = 2$ in the first quadrant. $(1,2)$

Implicit differentiation: $6x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = -\frac{3x + y}{x + y}$.

Gradient of tangent at $(1,2) = -\frac{3(1) + 2}{1 + 2} = -\frac{5}{3}$, gradient of normal at $(1,2) = \frac{3}{5}$.

Q3 Linearly dependent: Let $\tilde{c} = p\tilde{a} + q\tilde{b}$,

$m\tilde{i} + n\tilde{k} = (-3p - 2q)\tilde{i} + (2p - 2q)\tilde{j} + (3p + q)\tilde{k}$

$\therefore m = -3p - 2q$, $2p - 2q = 0$ and $n = 3p + q$

$\therefore p = q$, $m = -5p$ and $n = 4p$. $\therefore \frac{m}{n} = -\frac{5}{4}$.

Q4 $\sec\left(\frac{\pi}{5}\right) = \frac{1}{\cos\left(\frac{\pi}{5}\right)} = \frac{1}{\cos\left(2\left(\frac{\pi}{10}\right)\right)} = \frac{1}{1 - 2\sin^2\left(\frac{\pi}{10}\right)}$

$= \frac{1}{1 - 2\left(\frac{6 - 2\sqrt{5}}{16}\right)} = \frac{4}{\sqrt{5} + 1} = \frac{4\left(\sqrt{5} - 1\right)}{4} = \sqrt{5} - 1$

Q5a $v = -x^2$, $\frac{1}{2}v^2 = \frac{1}{2}x^4$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3$.

Initially $x = 1$ and $\therefore a = 2$.

Q5b $v = \frac{dx}{dt} = -x^2$, $\frac{dt}{dx} = -\frac{1}{x^2}$, $t = \int -\frac{1}{x^2}dx$, $t = \frac{1}{x} + c$.

When $t = 0$ (initially), $x = 1$, $\therefore c = -1$, $t = \frac{1}{x} - 1$.

Hence $x = \frac{1}{t + 1}$.

Q6 $f''(x) = -\sec^2(2x)$, $f'(x) = \int -\sec^2(2x)dx = -\frac{1}{2}\tan(2x) + c$.

$f'\left(\frac{\pi}{8}\right) = -\frac{1}{2}\tan\left(2\left(\frac{\pi}{8}\right)\right) + c = -1$, $\therefore -\frac{1}{2} + c = -1$, $c = -\frac{1}{2}$.

$\therefore f'(x) = -\frac{1}{2}\tan(2x) - \frac{1}{2}$.

The gradient at $x = \frac{\pi}{12}$ is

$f'\left(\frac{\pi}{12}\right) = -\frac{1}{2}\tan\left(2\left(\frac{\pi}{12}\right)\right) - \frac{1}{2} = -\frac{1}{2}\left(\frac{1}{\sqrt{3}} + 1\right) = -\frac{\sqrt{3} + 3}{6}$.

Q7 Horizontal component: $-T\sin 45° + 120\sin 30° = 0$,

$\therefore T = 60\sqrt{2}$.

Vertical component: $T\cos 45° + F - 120\cos 30° = 0$,

$\therefore F = 60\left(\sqrt{3} - 1\right)$.

Q8a $\overrightarrow{OA} = \tilde{i} + 5\tilde{k}$, $\overrightarrow{OB} = -\tilde{i} + 2\tilde{j} + 4\tilde{k}$, $\therefore \overrightarrow{AB} = -2\tilde{i} + 2\tilde{j} - \tilde{k}$

Q8b Let D be (p, q, r), $\overrightarrow{OD} = p\tilde{i} + q\tilde{j} + r\tilde{k}$.

$\overrightarrow{OC} = 3\tilde{i} + 5\tilde{j} + 2\tilde{k}$.

$\therefore \overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (3 - p)\tilde{i} + (5 - q)\tilde{j} + (2 - r)\tilde{k}$.

$ABCD$ is a parallelogram, $\therefore \overrightarrow{DC} = \overrightarrow{AB}$.

Hence $3 - p = -2$, $5 - q = 2$ and $2 - r = -1$.

$\therefore p = 5$, $q = 3$ and $r = 3$. $\therefore D(5,3,3)$.

Q8c $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 4\tilde{i} + 3\tilde{j} - 2\tilde{k}$

$\overrightarrow{AB}.\overrightarrow{BC} = -8 + 6 + 2 = 0$, $\overrightarrow{AB} \neq \tilde{0}$ and $\overrightarrow{BC} \neq \tilde{0}$,

$\therefore \overrightarrow{AB} \perp \overrightarrow{BC}$. $\therefore ABCD$ is a rectangle.

Q9a Area $= \pi \times 1 = \pi$ square units.

Q9b $y = \cos^{-1}(x)$, $\therefore x = \cos y$, $x^2 = \cos^2 y = \frac{1}{2}(\cos(2y)+1)$

$V = 2\int_0^{\frac{\pi}{2}} \pi x^2 dy = \pi \int_0^{\frac{\pi}{2}} (\cos(2y)+1) dy$

$= \pi \left[\frac{1}{2}\sin(2y) + y\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{2}$ cubic units.

Q10a $w = 1 + ai$, $|w| = (1+a^2)^{\frac{1}{2}}$,

$|w^3| = |w|^3 = \left((1+a^2)^{\frac{1}{2}}\right)^3 = (1+a^2)^{\frac{3}{2}}$.

Q10b $|w^3| = 8$, $|w|^3 = 8$, $|w| = 2$, $(1+a^2)^{\frac{1}{2}} = 2$,

$\therefore 1 + a^2 = 4$, $a^2 = 3$, $a = \pm\sqrt{3}$.

Q10c z that satisfy $|z^3| = 8$ are $z = 1 + i\sqrt{3}$, $z = 1 - i\sqrt{3}$, $z = 2$ and $z = -2$.

The 3 roots of $P(z) = 0$ must be 3 of the 4 above.

Since $P(z)$ has real coefficients, $\therefore P(z) = 0$ has a pair of complex conjugate roots and a real root, i.e.

$P(z) = (z - 1 - i\sqrt{3})(z - 1 + i\sqrt{3})(z - p)$, where $p = -2$ or 2.

Expand and collect like terms to obtain:

$P(z) = z^3 - (p+2)z^2 + (2p+4)z - 4p$.

Since b, c and d are non-zero real constants, $\therefore p + 2 \neq 0$, $2p + 4 \neq 0$ and $p \neq 0$, i.e. $p \neq -2$ and $p \neq 0$.

$\therefore p = 2$ is the only remaining possibility.

Hence $P(z) = z^3 - 4z^2 + 8z - 8$, $\therefore b = -4$, $c = 8$ and $d = -8$.

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