2008 VCAA Specialist Math Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
D	E	B	E	B	C	D	D	B	C	D

12	13	14	15	16	17	18	19	20	21	22
A	E	B	D	C	A	E	C	E	A	B

Q1 $y=\frac{x^4+a}{x^2}=x^2+\frac{a}{x^2}$, $a>0$. D

Q2 $x^2+ax+y^2+1=0$, $x^2+ax+\left(\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}-1$,

$\left(x+\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}-1$. To represent a circle, $\frac{a^2}{4}-1>0$,

$a^2>4$. $\therefore a<-2$ or $a>2$. E

Q3 $f(x)=3\sin^{-1}(4x-1)+\frac{\pi}{2}$ is an increasing function.

Domain: $-1\le 4x-1\le 1$, $0\le 4x\le 2$, $0\le x\le \frac{1}{2}$.

Range: $f(0)=3\sin^{-1}(-1)+\frac{\pi}{2}=-\pi$,

$f\left(\frac{1}{2}\right)=3\sin^{-1}(1)+\frac{\pi}{2}=2\pi$. $\therefore -\pi\le y\le 2\pi$ B

Q4 $m\in(-\infty,-2)\cup(2,\infty)$, i.e. $m\in R\setminus[-2,2]$. E

Q5 $\arg(z^7)=7Arg(z)=\frac{7\pi}{5}$, $\therefore Arg(z^7)=-\frac{3\pi}{5}$. B

Q6 $z=\frac{3+4i}{1+2i}=\frac{(3+4i)(1-2i)}{(1+2i)(1-2i)}=\frac{11}{5}+\left(-\frac{2}{5}\right)i$, $\mathrm{Im}(z)=-\frac{2}{5}$. C

Q7 $(z+2)(\bar{z}+2)=4$, $z\bar{z}+2(z+\bar{z})=0$. Let $z=x+iy$,
$x^2+y^2+4x=0$, $(x+2)^2+y^2=2^2$.
Radius is 2, centre is $(-2,0)$. D

Q8 $z=-1+i$, z is in the second quadrant.
$r=\sqrt{(-1)^2+1^2}=\sqrt{2}$, $\theta=\tan^{-1}\left(\frac{1}{-1}\right)=\frac{3\pi}{4}$.

$\therefore z=\sqrt{2}cis\left(\frac{3\pi}{4}\right)$. D

Q9 Choose the particular solution through O, $y=0.5\sin(2x)$.
$\frac{dy}{dx}=\cos(2x)$. B

Q10 $V=4h$, $\frac{dV}{dh}=4$.
$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $0.2-0.01\sqrt{h}=4\frac{dh}{dt}$.
$\therefore \frac{dh}{dt}=\frac{0.2-0.01\sqrt{h}}{4}=\frac{20-\sqrt{h}}{400}$. C

Q11 $y'=\frac{dy}{dx}=2\tan^{-1}(x+1)$

$x_0=0$ $y_0=1$ $y'(0)=2\tan^{-1}(1)=\frac{\pi}{2}$

$x_1=0.2$ $y_1=1+0.2\times\frac{\pi}{2}=1+0.1\pi$ $y'(1)=2\tan^{-1}(1.2)$

$x_2=0.4$ $y_2=1+0.1\pi+0.2\times 2\tan^{-1}(1.2)$
$=1+0.1\pi+0.4\tan^{-1}(1.2)$ D

Q12 The parabola is $y=f(x)=(x+3)(x-1)=x^2+2x-3$.
$\int_{-3}^{0}f(x)dx=\left[\frac{x^3}{3}+x^2-3x\right]_{-3}^{0}=-(-9+9+9)=-9$ A

Q13 $\tilde{r}(t)=15t\tilde{i}+(20t-5t^2)\tilde{j}$, $t\ge 0$.
$\tilde{v}(t)=\frac{d\tilde{r}}{dt}=15\tilde{i}+(20-10t)\tilde{j}$. At maximum height, $v_y=0$.
$\therefore 20-10t=0$, $t=2$, and $\tilde{r}(2)=30\tilde{i}+20\tilde{j}$. E

Q14 $\tilde{a}$ and $\tilde{b}$ are perpendicular, $\therefore \tilde{a}\cdot\tilde{b}=0$,
$m^2+4m-12=0$, $(m+6)(m-2)=0$, $m=-6$ or 2. B

Q15 $\tilde{P}=\tilde{i}$, $\tilde{Q}=a(\tilde{i}+\sqrt{3}\tilde{j})$, $|\tilde{Q}|=4$, $\therefore a\sqrt{1^2+(\sqrt{3})^2}=4$,
$\therefore a=2$ and $\tilde{Q}=2(\tilde{i}+\sqrt{3}\tilde{j})$.
$\tilde{P}+\tilde{Q}=\tilde{i}+2(\tilde{i}+\sqrt{3}\tilde{j})=3\tilde{i}+2\sqrt{3}\tilde{j}$,
$\therefore |\tilde{P}+\tilde{Q}|=\sqrt{3^2+(2\sqrt{3})^2}=\sqrt{21}$. D

Q16 Let $u=\tan^{-1}(x)$, $\frac{du}{dx}=\frac{1}{1+x^2}$.

When $x=0$, $u=0$; when $x=\sqrt{3}$, $u=\frac{\pi}{3}$.

$\int_0^{\sqrt{3}}\frac{\log_e(\tan^{-1}(x))}{1+x^2}dx=\int_0^{\sqrt{3}}\log_e(u)\frac{du}{dx}dx=\int_0^{\frac{\pi}{3}}\log_e(u)du$. C

Q17 $\left|\overrightarrow{QR}\right|=\frac{1}{2}\left|\overrightarrow{PQ}\right|$, $\therefore Q$ divides PR into a ratio of 2 : 1.
$\therefore \tilde{q}=\frac{\tilde{p}+2\tilde{r}}{3}$, $\therefore \tilde{r}=\frac{3}{2}\tilde{q}-\frac{1}{2}\tilde{p}$. A

Q18 Magnitude of $\tilde{F} = \tilde{F} \cdot \dfrac{\tilde{d}}{|\tilde{d}|}$. E

Comment: Wording problem? According to the information, $\tilde{F}$ causes the object to accelerate in the direction of $\tilde{d}$. $\therefore \tilde{F}$ and $\tilde{d}$ are in the same direction. If $\tilde{F}$ is known, then $|\tilde{F}|$ is the magnitude of $\tilde{F}$. Why would one want to find the magnitude of $\tilde{F}$ the long way?

Q19 $u = \dfrac{30}{5} = 6$, $t = 6$ and $v = \dfrac{40}{5} = 8$, use $s = \dfrac{1}{2}(u+v)t$ to find the displacement $s = \dfrac{1}{2}(6+8)6 = 42$ m.
Distance $= 42$ m. C

Q20 $v = \sin^{-1}(x)$, $a = v\dfrac{dv}{dx} = \sin^{-1}(x) \times \dfrac{1}{\sqrt{1-x^2}} = \dfrac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ E

Q21 $F_{friction} = 0.1 \times 10g = g$ newtons.
Resultant force driving the system $R = 4g - g = 3g$ newtons.
Acceleration $a = \dfrac{R}{m} = \dfrac{3g}{10+4} = \dfrac{3g}{14}$. A

Q22 $a = f(v)$, $\dfrac{dv}{dt} = f(v)$, $\dfrac{dt}{dv} = \dfrac{1}{f(v)}$, $t = \displaystyle\int \dfrac{1}{f(v)}dv$,

$t_1 = \displaystyle\int_{v_0}^{v_1} \dfrac{1}{f(v)}dv + t_0$. B

Section 2

Q1a $f(x) = \dfrac{6x\sqrt{x}}{3x^2+1}$, $x \in [0, \infty)$. Let $f'(x) = 0$ to locate the turning point(s). $\therefore 9\sqrt{x}(1-x^2) = 9\sqrt{x}(1-x)(1+x) = 0$,
$x = 0$ or 1.
$f''(1) = -1.125$ is a negative value.

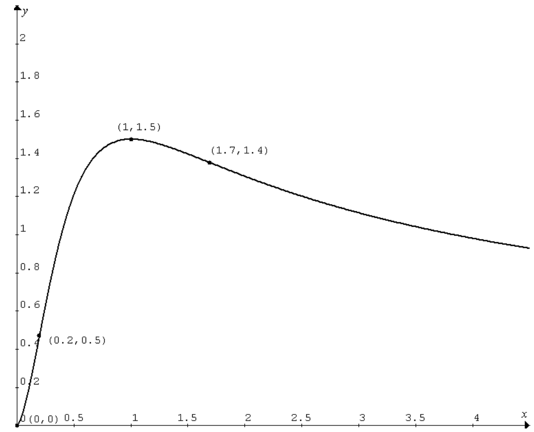
$\therefore$ the maximum turning point is at $x = 1$ and $y = f(1) = \dfrac{3}{2}$,

i.e. $\left(1, \dfrac{3}{2}\right)$.

Q1bi Let $f''(x) = 0$ to locate the inflection points.
$\therefore 9x^4 - 26x^2 + 1 = 0$.

Q1bii Use graphics calculator to solve for *x*, and to find *y*.
$x = 0.19745$, $y = 0.4713$ $(0.2, 0.5)$
$x = 1.688165$, $y = 1.3781$ $(1.7, 1.4)$.

Q1c



Q1di $y = \dfrac{6x\sqrt{x}}{3x^2+1}$, $y^2 = \dfrac{36x^3}{(3x^2+1)^2}$.

$V = \displaystyle\int_0^{\frac{1}{\sqrt{3}}} \pi y^2 dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \dfrac{18x^3}{(3x^2+1)^2}dx$.

Q1dii $u = 3x^2 + 1$, $3x^2 = u - 1$ and $\dfrac{du}{dx} = 6x$.

$V = 2\pi \displaystyle\int_0^{\frac{1}{\sqrt{3}}} \dfrac{18x^3}{(3x^2+1)^2}dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \dfrac{3x^2}{(3x^2+1)} \times 6x dx$

$= 2\pi \displaystyle\int_0^{\frac{1}{\sqrt{3}}} \dfrac{u-1}{u^2} \times \dfrac{du}{dx}dx = 2\pi \int_1^2 \left(\dfrac{u-1}{u^2}\right)du = 2\pi \int_1^2 \left(\dfrac{1}{u} - \dfrac{1}{u^2}\right)du$

Q1diii $V = 2\pi\left[\log_e u + \dfrac{1}{u}\right]_1^2 = 2\pi\left(\log_e 2 - \dfrac{1}{2}\right) = \pi(\log_e 4 - 1)$ cubic units.

Q2a $a = \dfrac{R}{m} = \dfrac{390-30}{80} = 4.5$ ms^{-2}.

Q2b $u = 0$, $s = 16$, $a = 4.5$. Use $v^2 = u^2 + 2as$ to find *v*.
$\therefore v = 12$. The speed is 12 ms^{-1}.

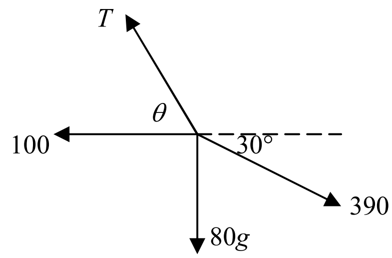
Q2c $a = \dfrac{R}{m} = \dfrac{390 - 30 - 6v}{80} = \dfrac{3}{40}(60 - v)$, where $v \geq 12$.

Q2d $\dfrac{dv}{dt} = \dfrac{3}{40}(60-v)$, $\dfrac{dt}{dv} = \dfrac{40}{3} \times \dfrac{1}{60-v}$, $t = \dfrac{40}{3}\displaystyle\int \dfrac{1}{60-v}dv$.
$\therefore \dfrac{3}{40}t = -\log_e(60-v) + c$.

When $t = 0$, $v = 12$. $\therefore c = \log_e 48$, and $t = \dfrac{40}{3}\log_e\left(\dfrac{48}{60-v}\right)$.

When $v = 18$, $t \approx 1.8$ s.

Q2ei



Q2eii Constant velocity, $\therefore$ zero resultant force.
Horizontal component: $390\cos 30° - T\cos\theta - 100 = 0$
Vertical component: $T\sin\theta - 390\sin 30° - 80g = 0$

Q2eiii $\cos\theta = \dfrac{390\cos 30° - 100}{T}$, $\sin\theta = \dfrac{390\sin 30° + 80g}{T}$.

$\therefore \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{390\sin 30° + 80g}{390\cos 30° - 100} = 4.118$.

Q2eiv $\theta = \tan^{-1}(4.118) = 76.35°$,

$T = \dfrac{390\sin 30° + 80g}{\sin 76.35°} \approx 1007$ N.

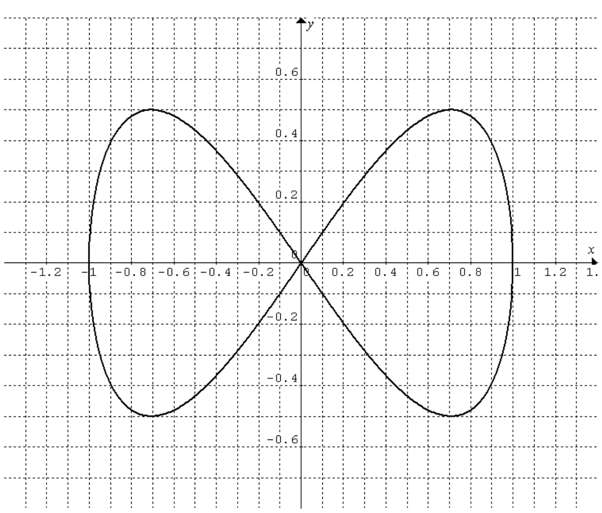
Q3ai $\tilde{r}(t) = \sin\left(\dfrac{t}{3}\right)\tilde{i} + \dfrac{1}{2}\sin\left(\dfrac{2t}{3}\right)\tilde{j}$, $t \ge 0$.

$y = \dfrac{1}{2}\sin\left(\dfrac{2t}{3}\right) = \sin\left(\dfrac{t}{3}\right)\cos\left(\dfrac{t}{3}\right)$, $\therefore y^2 = \sin^2\left(\dfrac{t}{3}\right)\cos^2\left(\dfrac{t}{3}\right)$.

Q3aii $\therefore y^2 = \sin^2\left(\dfrac{t}{3}\right)\left[1 - \sin^2\left(\dfrac{t}{3}\right)\right] = x^2\left(1 - x^2\right)$, where

$x = \sin\left(\dfrac{t}{3}\right)$.

Q3b



Q3c $x = \sin\left(\dfrac{t}{3}\right)$, period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$.

Q3d $\tilde{v}(t) = \dfrac{d\tilde{r}}{dt} = \dfrac{1}{3}\cos\left(\dfrac{t}{3}\right)\tilde{i} + \dfrac{1}{3}\cos\left(\dfrac{2t}{3}\right)\tilde{j}$.

Speed $= \left|\tilde{v}(t)\right| = \dfrac{1}{3}\sqrt{\cos^2\left(\dfrac{t}{3}\right) + \cos^2\left(\dfrac{2t}{3}\right)}$

The train passes through the origin at $t = 0$, 3π, 6π, …

$\therefore$ speed $= \dfrac{\sqrt{2}}{3}$ ms^{-1}.

Q3ei Distance $= 4\int_0^{1.5\pi}\left|\tilde{v}(t)\right|dt = \dfrac{4}{3}\int_0^{1.5\pi}\sqrt{\cos^2\left(\dfrac{t}{3}\right) + \cos^2\left(\dfrac{2t}{3}\right)}\,dt$.

Q3eii By graphics calculator: Distance ≈ 6.1 m.

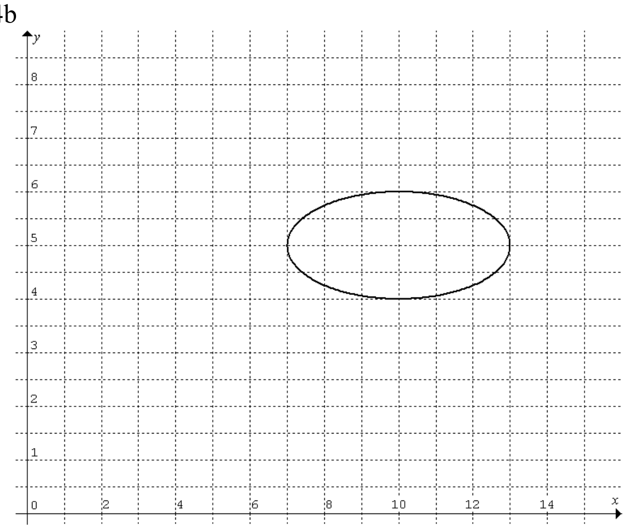
Q4a Rabbits: $x = 10 + 3\cos\left(\dfrac{\pi t}{6}\right)$, $t \ge 0$.

Foxes: $y = 5 + \sin\left(\dfrac{\pi t}{6}\right)$, $t \ge 0$.

$\cos\left(\dfrac{\pi t}{6}\right) = \dfrac{x - 10}{3}$, $\sin\left(\dfrac{\pi t}{6}\right) = y - 5$.

$\cos^2\left(\dfrac{\pi t}{6}\right) + \sin^2\left(\dfrac{\pi t}{6}\right) = 1$, $\therefore \dfrac{(x-10)^2}{9} + (y-5)^2 = 1$.

Q4b



Q4ci $x_{\min} = 7$ when $\cos\left(\dfrac{\pi t}{6}\right) = -1$. $\dfrac{\pi t}{6} = \pi$, $t = 6$ months.

Q4cii When $t = 6$, $y = 5$, i.e. 500 foxes.

Q4di $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{-0.2y + 0.02xy}{0.5x - 0.1xy} = \dfrac{xy - 10y}{25x - 5xy}$.

Q4dii $25\log_e(y)-5y-x+10\log_e(x)=c$

Implicit differentiation: $\frac{25}{y}\frac{dy}{dx}-5\frac{dy}{dx}-1+\frac{10}{x}=0$,

$\left(\frac{25}{y}-5\right)\frac{dy}{dx}=1-\frac{10}{x}$,

$\frac{dy}{dx}=\frac{1-\frac{10}{x}}{\frac{25}{y}-5}=\frac{1-\frac{10}{x}}{\frac{25}{y}-5}\times\frac{xy}{xy}=\frac{xy-10y}{25x-5xy}$.

Q4e As $x\to x_{\min}$ or $x_{\max}$, $\frac{dy}{dx}\to\infty$, $\frac{dx}{dy}\to 0$.

$\therefore \frac{25x-5xy}{xy-10y}\to 0$, where $x,y>0$.

$\therefore 25x-5xy\to 0$, $y\to 5$.

Let $y=5$, $25\log_e(5)-5(5)-x+10\log_e(x)=27.5$,

$25\log_e(5)-x+10\log_e(x)=52.5$.

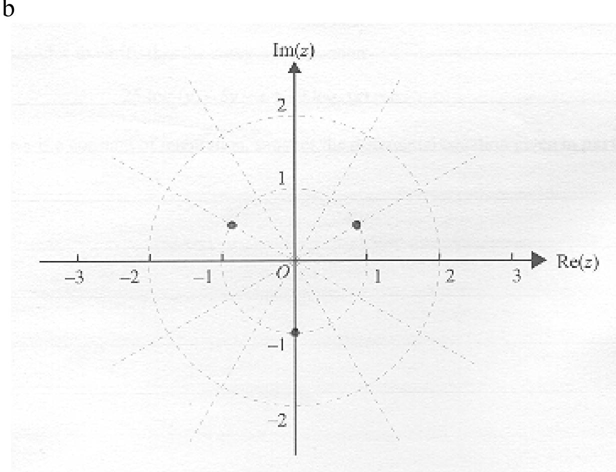
By graphics calculator: $x_{\min}=6.5871$, $x_{\max}=14.4269$.

Minimum number of rabbits ≈ 6590.

Maximum number of rabbits ≈ 14430.

Q5a $z=\frac{\sqrt{3}}{2}+\frac{1}{2}i=cis\left(\frac{\pi}{6}\right)$, $z^3=cis\left(3\times\frac{\pi}{6}\right)=cis\left(\frac{\pi}{2}\right)=i$.

Q5b



Q5c $|z-i|=1$ is a circle: $x^2+(y-1)^2=1$.

$\text{Re}(z)=-\frac{1}{\sqrt{3}}\text{Im}(z)$ is a straight line: $y=-\sqrt{3}x$.

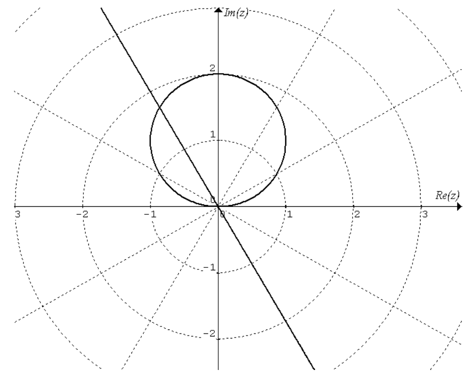
Solve the two equations simultaneously to find the coordinates of the points of intersection.

$x^2+\left(-\sqrt{3}x-1\right)^2=1$, expand and simplify to $4x^2+2\sqrt{3}x=0$.

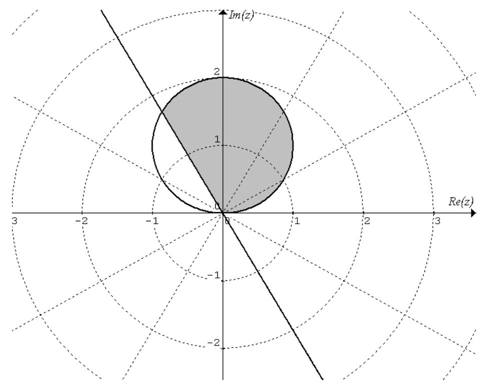
$\therefore x=0$ and $y=0$, or $x=-\frac{\sqrt{3}}{2}$ and $y=\frac{3}{2}$.

The two points are $z=0$, $z=-\frac{\sqrt{3}}{2}+\frac{3}{2}i$.

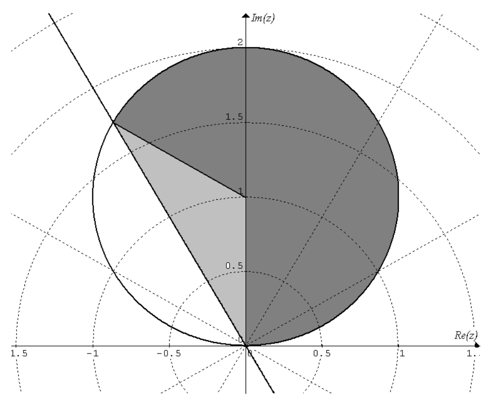
Q5d



Q5e



Q5f



Area of the dark shaded region $=\frac{2}{3}$ of the area of the circle

$=\frac{2}{3}\pi$.

Area of the light shaded region $=\frac{1}{2}\times 1\times 1\times\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{4}$.

Total area $=\frac{2\pi}{3}+\frac{\sqrt{3}}{4}\approx 2.53$ square units.

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