2009 NSW BOS Mathematics Extension 1 Solutions

Q1a $8x^3+27=(2x)^3+3^3=(2x+3)(4x^2-6x+9)$

Q1b $f(x)=\ln(x-3)$, $x-3>0$, $x>3$. Domain is $(3,\infty)$.

Q1c $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}=2\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cos x\to 2\times 1=2$.

Q1d $\dfrac{x+3}{2x}>1$, $\dfrac{1}{2}+\dfrac{3}{2x}>1$, $\dfrac{3}{2x}>\dfrac{1}{2}$, $\dfrac{1}{x}>\dfrac{1}{3}$, $\therefore 0<x<3$.

Q1e

$\dfrac{d}{dx}\left(x\cos^2 x\right)=(1)\cos^2 x+x(-2\cos x\sin x)=\cos^2 x-x\sin 2x$.

Q1f $u=x^3+1$, $\dfrac{1}{3}\dfrac{du}{dx}=x^2$.

$$\int_0^2 x^2e^{x^3+1}dx=\int_0^2\frac{1}{3}e^u\frac{du}{dx}dx=\int_1^9\frac{1}{3}e^u du=\left[\frac{1}{3}e^u\right]_1^9=\frac{e}{3}\left(e^8-1\right).$$

Q2a $p(x)=x^3-ax+b$.

Divided by $x-1$, $R=p(1)=1-a+b=2$, $\therefore -a+b=1$ ……(1)

Divided by $x+2$, $R=p(-2)=-8+2a+b=5$,

$\therefore 2a+b=13$ ……(2)

(2) − (1): $3a=12$, $\therefore a=4$ and $b=5$.

Q2bi Let $3\sin x+4\cos x=A\sin(x+\alpha)$.

$\therefore 3\sin x+4\cos x=A\sin x\cos\alpha+A\cos x\sin\alpha$.

Hence, $A\cos\alpha=3$ ……(1)

and $A\sin\alpha=4$ ……(2)

$\therefore (A\cos\alpha)^2+(A\sin\alpha)^2=25$, $A^2\left(\cos^2\alpha+\sin^2\alpha\right)=25$, $A=5$.

$\therefore \sin\alpha=\dfrac{4}{5}$, $\alpha=\sin^{-1}0.8$.

Hence $3\sin x+4\cos x=5\sin\left(x+\sin^{-1}0.8\right)$.

Q2bii $3\sin x+4\cos x=5$, $5\sin\left(x+\sin^{-1}0.8\right)=5$,

$\sin\left(x+\sin^{-1}0.8\right)=1$.

$\therefore x+\sin^{-1}0.8=\dfrac{\pi}{2}$, $x\approx 0.64$.

Q2ci At $P\left(2t,t^2\right)$, $\dfrac{dy}{dx}=\dfrac{x}{2}=t$.

Equation of tangent at $\left(2t,t^2\right)$: $y-t^2=t(x-2t)$,

$\therefore y=tx-t^2$ ……(1)

Q2cii At $Q\left(4t,4t^2\right)$, i.e. $Q\left(2(2t),(2t)^2\right)$, equation of tangent is

$y=(2t)x-(2t)^2$, $\therefore y=2tx-4t^2$ ……(2)

Solve (1) and (2) simultaneously to find intersection *R*.

$\therefore 2tx-4t^2=tx-t^2$, $tx-3t^2=0$, $t(x-3t)=0$.

For the situation as shown in the diagram, $t\neq 0$, $\therefore x=3t$ and $y=2t^2$. Hence $R\left(3t,2t^2\right)$, where $t\neq 0$.

Note: No unique intersection when $t=0$.

Q2ciii $R\left(3t,2t^2\right)$, $x=3t$, $\therefore t=\dfrac{x}{3}$.

$y=2t^2$, $\therefore y=2\left(\dfrac{x}{3}\right)^2$, i.e. $y=\dfrac{2}{9}x^2$, $x\neq 0$.

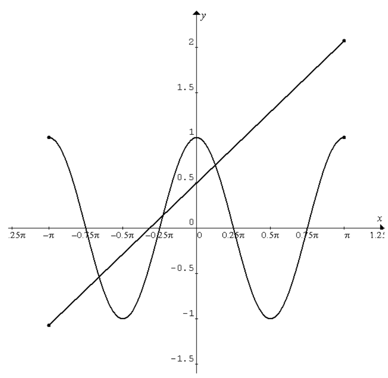
Q3ai $f(x)=\dfrac{3+e^{2x}}{4}=\dfrac{1}{4}e^{2x}+\dfrac{3}{4}$. The range is $\left(\dfrac{3}{4},\infty\right)$.

Q3aii Equation of inverse: $x=\dfrac{3+e^{2y}}{4}$, $4x-3=e^{2y}$,

$y=\dfrac{1}{2}\ln(4x-3)$ or $y=\ln\sqrt{4x-3}$.

$\therefore f^{-1}(x)=\ln\sqrt{4x-3}$.

Q3bi



Q3bii $2\cos 2x=x+1$, $\cos 2x=\dfrac{x+1}{2}$. The solutions to this equation are the *x*-coordinates of the intersections of the graphs of $y=\cos 2x$ and $y=\dfrac{x+1}{2}$. There are 3 intersections, ∴ 3 solutions.

Q3biii $f(x)=2\cos 2x-(x+1)$, $f'(x)=-4\sin 2x-1$.

Newton's method: $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$.

$x_1=0.4$

$x_2=0.4-\dfrac{2\cos 0.8-1.4}{-4\sin 0.8-1}\approx 0.398$.

Q3ci $R.H.S. = \frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{1-\left(1-2\sin^2\theta\right)}{1+\left(2\cos^2\theta - 1\right)}$

$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = L.H.S.$

Q3cii $\tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$,

$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{\left(\sqrt{2}-1\right)\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}}$

$= \sqrt{\left(\sqrt{2}-1\right)^2} = \sqrt{2}-1$.

Q4ai Binomial: $n = 5$, $p = \frac{1}{4}$, $q = \frac{3}{4}$.

$\Pr(X = 3) = \binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{45}{512}$.

Q4aii $\Pr(X \ge 3) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$

$= \binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 + \binom{5}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0 = \frac{53}{512}$.

Q4aiii Let X' be the number of incorrect answers.

$\Pr(X' \ge 1) = 1 - \Pr(X' = 0) = 1 - \binom{5}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 = \frac{1023}{1024}$.

Q4bi $f(x) = \frac{x^4 + 3x^2}{x^4 + 3}$.

$f(-x) = \frac{(-x)^4 + 3(-x)^2}{(-x)^4 + 3} = \frac{x^4 + 3x^2}{x^4 + 3} = f(x)$.

$\therefore f(x)$ is an even function.

Q4bii $f(x) = \frac{x^4 + 3x^2}{x^4 + 3} = 1 + \frac{3x^2 - 3}{x^4 + 3}$.

Horizontal asymptote is $y = 1$.

Q4biii $f(x) = 1 + \frac{3x^2 - 3}{x^4 + 3}$,

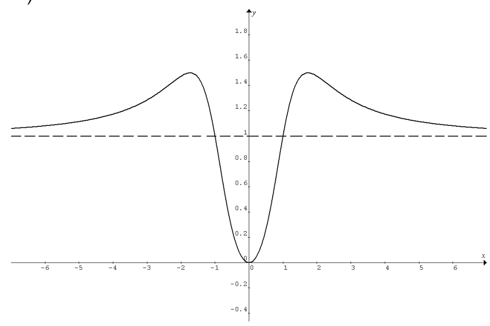
$f'(x) = \frac{\left(x^4+3\right)(6x) - \left(3x^2 - 3\right)\left(4x^3\right)}{\left(x^4+3\right)^2} = \frac{18x + 12x^3 - 6x^5}{\left(x^4+3\right)^2} = 0$.

$\therefore 18x + 12x^3 - 6x^5 = 0$, $6x\left(3 + 2x^2 - x^4\right) = 0$,

$6x\left(3 - x^2\right)\left(1 + x^2\right) = 0$, $6x\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right)\left(1 + x^2\right) = 0$.

Since $1 + x^2 \ne 0$, $\therefore x = 0$, $\pm\sqrt{3}$ are the *x*-coordinates of the stationary points.

Q4biv When $x = 0$, $y = f(0) = 0$. When $x = \pm\sqrt{3}$, $y = f\left(\pm\sqrt{3}\right) = 1.5$.



Q5ai $\frac{d^2x}{dt^2} = -n^2x$, or $\frac{1}{2}\frac{d\left(v^2\right)}{dx} = -n^2x$, $\frac{d\left(v^2\right)}{dx} = -2n^2x$,

$v^2 = \int -2n^2x\,dx = -n^2x^2 + c$.

At $x = a$, $v = 0$. $\therefore c = n^2a^2$ and $v^2 = n^2\left(a^2 - x^2\right)$.

Q5aii Speed is maximum at $x = 0$, $v^2 = n^2a^2$, $\therefore |v|_{\max} = na$.

Q5aiii |Acceleration| is maximum at $x = a$, $\therefore \left|\frac{d^2x}{dt^2}\right|_{\max} = n^2a$.

Q5aiv Consider $v = n\sqrt{a^2 - x^2}$. Note: $v = -n\sqrt{a^2 - x^2}$ will yield the same result.

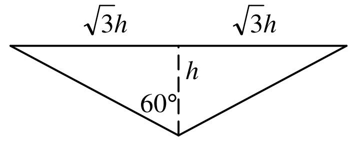
$\frac{dx}{dt} = n\sqrt{a^2 - x^2}$, $\frac{dt}{dx} = \frac{1}{n\sqrt{a^2 - x^2}}$, $t = \int \frac{1}{n\sqrt{a^2 - x^2}}dx$,

$nt = \sin^{-1}\frac{x}{a} + c$. When $t = 0$, $x = 0$, $\therefore c = 0$ and $nt = \sin^{-1}\frac{x}{a}$.

Hence $x = a\sin nt$.

Now $v = \frac{dx}{dt} = an\cos nt = \frac{na}{2}$, $\cos nt = \frac{1}{2}$, $nt = \frac{\pi}{3}$, $\therefore t = \frac{\pi}{3n}$ is the first time when the particle's speed is half its maximum speed.

Q5bi



Area of cross-section $= \frac{1}{2} \times 2\sqrt{3}h \times h = \sqrt{3}h^2$ m².

Volume of water $V = 10\sqrt{3}h^2$ m³.

Q5bii Top water surface area $A = 10 \times 2\sqrt{3}h = 20\sqrt{3}h$ m².

Q5biii $\frac{dV}{dt}=-kA$, $\frac{dV}{dh}\times\frac{dh}{dt}=-kA$, $20\sqrt{3}h\frac{dh}{dt}=-k\times20\sqrt{3}h$,

$\therefore\frac{dh}{dt}=-k$, where $k>0$. Note that the depth of water decreases at a constant rate.

Q5biv The constant rate is 1 m per 100 days. $\therefore$ it takes another 100 days to fall from 2 m to 1 m.

Q6ai At time $t=T$, $x_1=x_2$. $\therefore UT\cos\theta=R-VT\cos\theta$,
$UT\cos\theta+VT\cos\theta=R$, $T(U+V)\cos\theta=R$,

$\therefore T=\frac{R}{(U+V)\cos\theta}$.

Q6aii At time $t=T$,
$y_1=UT\sin\theta-\frac{1}{2}gT^2$ and $y_2=h-VT\sin\theta-\frac{1}{2}gT^2$.

$h-VT\sin\theta=R\tan\theta-VT\sin\theta=\frac{R\sin\theta}{\cos\theta}-\frac{VR\sin\theta}{(U+V)\cos\theta}$

$=\frac{(U+V)R\sin\theta-VR\sin\theta}{(U+V)\cos\theta}=\frac{UR\sin\theta}{(U+V)\cos\theta}=UT\sin\theta$.

$\therefore y_1=y_2$. Hence the projectiles collide.

Q6aiii At time $t=T$, $x=\lambda R$ and $x=UT\cos\theta$, where $0<\lambda<1$.

$\therefore\lambda R=UT\cos\theta$, $\lambda R=\frac{UR}{U+V}$, $\lambda=\frac{U}{U+V}$, $\lambda U+\lambda V=U$.

Hence $V=\frac{U-\lambda U}{\lambda}=\left(\frac{1-\lambda}{\lambda}\right)U=\left(\frac{1}{\lambda}-1\right)U$.

Q6bi

$(1+x)^r+(1+x)^{r+1}+...+(1+x)^n=(1+x)^r\left[1+(1+x)+...+(1+x)^{n-r}\right]$

$=(1+x)^r\left[\frac{1\left((1+x)^{n-r+1}-1\right)}{(1+x)-1}\right]=(1+x)^r\left[\frac{(1+x)^{n-r+1}-1}{x}\right]$

$=\frac{(1+x)^{n+1}-(1+x)^r}{x}$.

$\therefore(1+x)^r+(1+x)^{r+1}+...+(1+x)^n=\frac{(1+x)^{n+1}}{x}-\frac{(1+x)^r}{x}$

Consider only the coefficients of the x^r terms in the expression above: $\binom{r}{r}+\binom{r+1}{r}+...+\binom{n}{r}=\binom{n+1}{r+1}$.

Note that for $(1+x)^{n+1}$, the coefficient of x^{r+1} is $\binom{n+1}{r+1}$.

$\therefore$ for $\frac{(1+x)^{n+1}}{x}$, the coefficient of x^r is $\binom{n+1}{r+1}$. Also $\frac{(1+x)^r}{x}$ does not have an x^r term.

Q6bii(1) There are n points on the line $y=x$. Select any 2 points to form an interval, the number of intervals is $\binom{n}{2}$.

Q6bii(2) For 1 point, number of intervals is 0.

For 2 points, number of intervals is $\binom{2}{2}$.

For 3 points, number of intervals is $\binom{3}{2}$.

For n–1 points, number of intervals is $\binom{n-1}{2}$.

For n points, number of intervals is $\binom{n}{2}$.

$\therefore S_n=\binom{2}{2}+\binom{3}{2}+...+\binom{n-1}{2}+\binom{n}{2}+\binom{n-1}{2}+...+\binom{3}{2}+\binom{2}{2}$.

Q6biii $\therefore S_n=2\left[\binom{2}{2}+\binom{3}{2}+...+\binom{n-1}{2}\right]+\binom{n}{2}$.

Using the result in Q6bi, $S_n=2\binom{n-1+1}{2+1}+\binom{n}{2}=2\binom{n}{3}+\binom{n}{2}$

$=2\times\frac{n(n-1)(n-2)}{3\times2\times1}+\frac{n(n-1)}{2\times1}=\frac{2n(n-1)(n-2)+3n(n-1)}{6}$

$=\frac{n(n-1)[2(n-2)+3]}{6}=\frac{n(n-1)(2n-1)}{6}$.

Q7ai $\lim_{h\to0}\frac{(x+h)-x}{h}=\lim_{h\to0}1=1$.

Q7aii $\frac{d}{dx}(x)=1$, $\therefore$ the statement is true for $n=1$.

Assume it is true for $n=k$, i.e. $\frac{d}{dx}\left(x^k\right)=kx^{k-1}$.

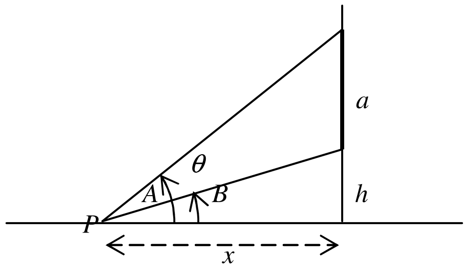
Now consider $n=k+1$.

$\frac{d}{dx}\left(x^{k+1}\right)=\frac{d}{dx}\left(x.x^k\right)=(x)\left(kx^{k-1}\right)+(1)\left(x^k\right)=(k+1)x^k$.

$\therefore$ the statement is true for $n=k+1$.

$\therefore$ the statement is true for all positive integers n.

Q7bi



From the diagram, $\theta = A - B$, $\tan A = \dfrac{a+h}{x}$, $\tan B = \dfrac{h}{x}$.

$$\tan\theta = \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} = \frac{\frac{a+h}{x} - \frac{h}{x}}{1+\frac{a+h}{x}\times\frac{h}{x}} = \frac{\frac{a}{x}}{1+\frac{h(a+h)}{x^2}}$$

$= \dfrac{ax}{x^2+h(a+h)}$. Hence $\theta = \tan^{-1}\left[\dfrac{ax}{x^2+h(a+h)}\right]$.

Q7bii $\tan\theta = \dfrac{ax}{x^2+h(a+h)}$, $\dfrac{d}{dx}(\tan\theta) = \dfrac{d}{dx}\left[\dfrac{ax}{x^2+h(a+h)}\right]$,

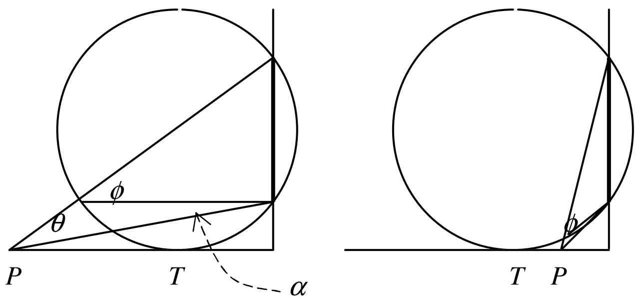
$\sec^2\theta \times \dfrac{d\theta}{dx} = \dfrac{(x^2+h(a+h))(a)-(ax)(2x)}{(x^2+h(a+h))^2}$,

$\dfrac{d\theta}{dx} = \dfrac{a(x^2+h(a+h))-2ax^2}{(x^2+h(a+h))^2\sec^2\theta}$.

$\dfrac{d\theta}{dx} = 0$ when $a(x^2+h(a+h))-2ax^2 = 0$.

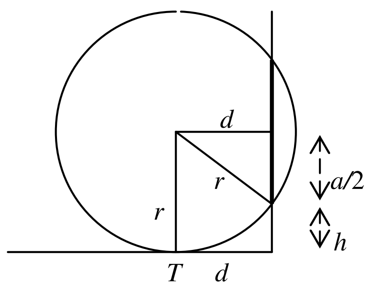
$\therefore x^2 = h(a+h)$ and $x = \sqrt{h(a+h)}$ for maximum θ.

Q7ci



In both cases, $\theta + \alpha = \phi$, $\therefore \theta < \phi$ when *P* and *T* are different points. When *P* and *T* are the same point, $\theta = \phi$ is a maximum.

Q7cii



Radius of the circle $r = \dfrac{a}{2} + h$.

Distance of *T* from the building :

$$d = \sqrt{\left(\frac{a}{2}+h\right)^2 - \left(\frac{a}{2}\right)^2} = \sqrt{h(a+h)}.$$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.