2009 NSW BOS Mathematics Extension 2 Solutions
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Q1a Let $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$.

$\int \frac{\ln x}{x} dx = \int u \frac{du}{dx} dx = \int u du = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln x)^2 + c$.

Q1b Let $u = x$ and $v = \frac{1}{2}e^{2x}$, $\frac{du}{dx} = 1$, $\frac{dv}{dx} = e^{2x}$.

$\int xe^{2x} dx = \int u \frac{dv}{dx} dx = \int u dv = uv - \int v du$

$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\frac{du}{dx}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$.

Q1c $\int \frac{x^2}{1+4x^2}dx = \frac{1}{4}\int \frac{4x^2}{1+4x^2}dx = \frac{1}{4}\int \left(1 - \frac{1}{1+4x^2}\right)dx$

$= \frac{1}{4}\left(x - \frac{1}{2}\tan^{-1}(2x)\right) + c = \frac{1}{4}x - \frac{1}{8}\tan^{-1}(2x) + c$.

Q1d $\int_2^5 \frac{x-6}{x^2+3x-4}dx = \int_2^5 \left(\frac{2}{x+4} - \frac{1}{x-1}\right)dx$ [Partial fractions]

$= [2\ln(x+4) - \ln(x-1)]_2^5 = \left[\ln\frac{(x+4)^2}{x-1}\right]_2^5 = \ln\frac{9}{16}$.

Q1e Let $x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$, $\frac{d\theta}{dx} = \frac{1}{\sec^2\theta}$.

$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}dx = \int_1^{\sqrt{3}} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}dx = \int_1^{\sqrt{3}} \frac{1}{\tan^2\theta\sec\theta}dx$

$\int_1^{\sqrt{3}} \frac{\sec\theta}{\tan^2\theta}\frac{d\theta}{dx}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u^2}\frac{du}{d\theta}d\theta = \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^2}du$

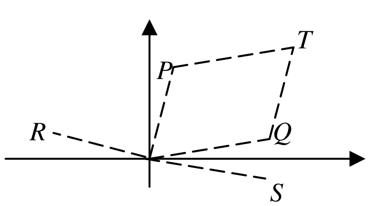
Let $u = \sin\theta$, $\frac{du}{d\theta} = \cos\theta$

$= \left[-\frac{1}{u}\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \frac{3\sqrt{2} - 2\sqrt{3}}{3}$.

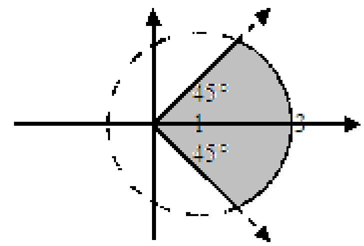
Q2a $i^9 = (i^4)^2 i = i$.

Q2b $\frac{-2+3i}{2+i} \times \frac{2-i}{2-i} = \frac{-1+8i}{5} = -\frac{1}{5} + \frac{8}{5}i$.

Q2c



Q2d

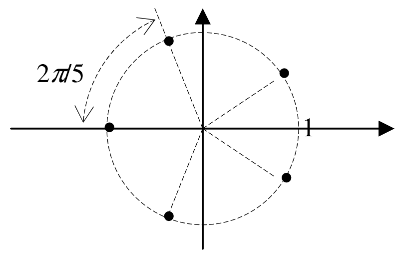


Q2ei $z^5 = -1 = cis((2n+1)\pi)$, $\therefore z = cis\left(\frac{2n+1}{5}\pi\right)$.

Let $n = 0$, ± 1, ± 2.

$z = cis\left(-\frac{3\pi}{5}\right)$, $cis\left(-\frac{\pi}{5}\right)$, $cis\left(\frac{\pi}{5}\right)$, $cis\left(\frac{3\pi}{5}\right)$, $cis(\pi)$.

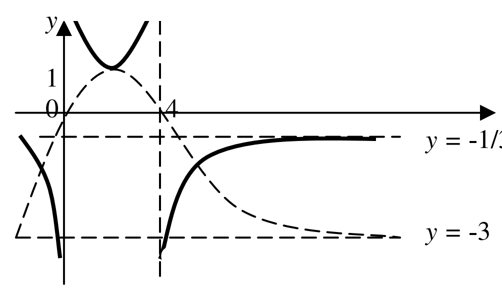
Q2eii



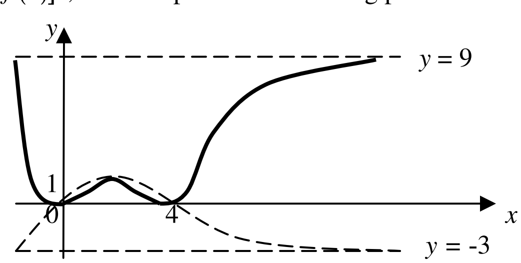
Q2fi Let $a + bi = \sqrt{3+4i}$.
$\therefore a^2 - b^2 = 3$ and $ab = 2$.
Hence $a = \pm 2$ and $b = \pm 1$.
The two square roots are $\pm(2+i)$.

Q2fii $z = \frac{-i \pm \sqrt{i^2 - 4(-1-i)}}{2} = \frac{-i \pm \sqrt{3+4i}}{2} = \frac{-i \pm (2+i)}{2}$.
$\therefore z = 1$ or $-1-i$.

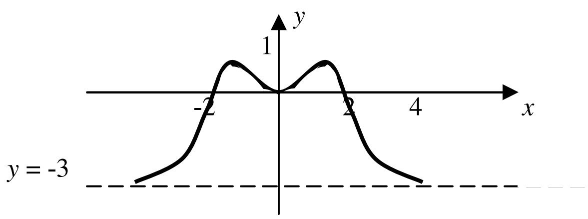
Q3ai $y = \frac{1}{f(x)}$, undefined at $x = 0$ and $x = 4$.



Q3aii $y = [f(x)]^2$, x-intercepts become turning points.



Q3aiii $y = f(x^2)$, $f((-x)^2) = f(x^2)$, $\therefore$ symmetrical about the y-axis. $f(2)$ is a local maximum value approximately, $\therefore x^2 \approx 2$, $\therefore x \approx \pm\sqrt{2}$. $\frac{d}{dx} f(x^2) = 2x \frac{d}{d(x^2)} f(x^2)$, at $x = 0$, $\frac{d}{dx} f(x^2) = 0$.



Q3bi $x^2 + 2xy + 3y^2 = 18$,

$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ [implicit differentiation],

$\frac{dy}{dx} = -\frac{x+y}{x+3y}$.

For horizontal tangent, $\frac{dy}{dx} = -\frac{x+y}{x+3y} = 0$,

$\therefore x + y = 0$, i.e. $y = -x$.

$\therefore x^2 + 2x(-x) + 3(-x)^2 = 18$, $2x^2 = 18$, $\therefore x = \pm 3$ and $y = \mp 3$.

The points are $(3,-3)$ and $(-3,3)$.

Q3c $P(x) = x^3 + ax^2 + bx + 5 = (x-1)^2(x+5) = x^3 + 3x^2 - 9x + 5$.

$\therefore a = 3$ and $b = -9$.

Q3d Solve $y = x + 1$ and $y = (x-1)^2$ simultaneously.

$x + 1 = (x-1)^2$, $3x - x^2 = 0$, $\therefore x = 0$ or 3.

$V = \int_0^3 2\pi x\left(x + 1 - (x-1)^2\right)dx = 2\pi\int_0^3 \left(3x^2 - x^3\right)dx$

$= 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^3 = \frac{27\pi}{2}$ cubic units.

Q4ai $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, implicit differentiation, $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$.

Gradient of tangent $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$,

and hence gradient of the normal at (x_0, y_0) is $\frac{a^2y_0}{b^2x_0}$.

Equation of the normal is $y - y_0 = \frac{a^2y_0}{b^2x_0}(x - x_0)$.

Q4aii x-intercept of the normal: Let $y = 0$, $-y_0 = \frac{a^2y_0}{b^2x_0}(x - x_0)$,

$\therefore x = x_0 - \frac{b^2x_0}{a^2} = \left(1 - \frac{b^2}{a^2}\right)x_0 = e^2x_0$.

Hence N is $(e^2x_0, 0)$.

Q4aiii $PS = ePM$, $PS' = ePM'$.

$\therefore \frac{PS}{PS'} = \frac{ePM}{ePM'} = \frac{e\left(\frac{a}{e} - x_0\right)}{e\left(\frac{a}{e} + x_0\right)} = \frac{a - ex_0}{a + ex_0} = \frac{ae - e^2x_0}{ae + e^2x_0} = \frac{NS}{NS'}$.

Q4aiv $\frac{\sin\alpha}{NS'} = \frac{\sin\angle PNS'}{PS'}$, $\therefore \sin\alpha = \frac{NS'\sin\angle PNS'}{PS'}$.

$\frac{\sin\beta}{NS} = \frac{\sin\angle PNS}{PS}$, $\therefore \sin\beta = \frac{NS\sin\angle PNS}{PS}$.

$\therefore \frac{\sin\alpha}{\sin\beta} = \frac{PS.NS'\sin\angle PNS'}{PS'.NS\sin\angle PNS}$.

$\angle PNS'$ and $\angle PNS$ are supplementary angles,

$\therefore \sin\angle PNS' = \sin\angle PNS$.

From Q4aiii, $\frac{PS.NS'}{PS'.NS} = 1$.

$\therefore \frac{\sin\alpha}{\sin\beta} = 1$, i.e. $\sin\alpha = \sin\beta$.

$\therefore \alpha = \beta$ since $\alpha + \beta < \pi$.

Q4bi $0 < \alpha < \frac{\pi}{2}$.

Vertical: $N\sin\alpha + T\cos\alpha - mg = 0$ ……(1)

Horizontal: $T\sin\alpha - N\cos\alpha = mr\omega^2$ ……(2)

Q4bii From equation (1), $N = \frac{mg - T\cos\alpha}{\sin\alpha}$ ……(3)

Substitute in equation (2), $T\sin\alpha - \frac{(mg - T\cos\alpha)\cos\alpha}{\sin\alpha} = mr\omega^2$.

Simplify and write T as the subject, $T = m\left(g\cos\alpha + r\omega^2\sin\alpha\right)$.

Substitute in equation (3) and simplify,

$N = \frac{mg - m\left(g\cos\alpha + r\omega^2\sin\alpha\right)\cos\alpha}{\sin\alpha} = m\left(g\sin\alpha - r\omega^2\cos\alpha\right)$.

Q4biii If $T = N$,

$m\left(g\cos\alpha + r\omega^2\sin\alpha\right) = m\left(g\sin\alpha - r\omega^2\cos\alpha\right)$,

$r\omega^2(\sin\alpha + \cos\alpha) = g(\sin\alpha - \cos\alpha)$,

$\omega^2 = \frac{g(\sin\alpha - \cos\alpha)}{r(\sin\alpha + \cos\alpha)} = \frac{g}{r}\left(\frac{\frac{\sin\alpha - \cos\alpha}{\cos\alpha}}{\frac{\sin\alpha + \cos\alpha}{\cos\alpha}}\right) = \frac{g}{r}\left(\frac{\tan\alpha - 1}{\tan\alpha + 1}\right)$ ……(4)

Q4biv Solve $\omega^2 = \frac{g}{r}\left(\frac{\tan\alpha - 1}{\tan\alpha + 1}\right)$ for α in terms of r and ω,

$\alpha = \tan^{-1}\left(\frac{g + r\omega^2}{g - r\omega^2}\right)$.

Since $\frac{g + r\omega^2}{g - r\omega^2} > 1$, $\therefore \alpha > \frac{\pi}{4}$.

Hence $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

Q5ai Since AB is a diameter, $\therefore \angle ADB = 90°$.
ΔAKY and ΔABD are similar because $\angle A$ is common and $\angle AYK = \angle ADB = 90°$. Hence $\angle AKY = \angle ABD$.

Q5aii $\angle ABD = \angle ACD$ [Subtended by the same arc *AD*].
$\angle DKX = \angle ABD$ [From Q5ai].
$\therefore \angle DKX = \angle DCX$.
$\therefore CKDX$ is a cyclic quadrilateral because both angles are subtended by the same arc *DX*.

Q5aiii *KX* is a diameter since $\angle XDK = 90°$. $\therefore \angle XCK = 90°$.
$\angle ACB = \angle ADB = 90°$ [Subtended by diameter *AB*].
$\therefore \angle XCK$ and $\angle ACB$ form a straight angle.
Hence *B*, *C* and *K* are collinear.

Q5bi $I_n = \int_0^1 x^{2n+1} e^{x^2} dx$, $\therefore I_{n-1} = \int_0^1 x^{2(n-1)+1} e^{x^2} dx = \int_0^1 x^{2n-1} e^{x^2} dx$.

Let $u = x^{2n}$, $\frac{du}{dx} = 2nx^{2n-1}$, and $\frac{dv}{dx} = xe^{x^2}$, $v = \frac{1}{2}e^{x^2}$.

$I_n = [u(x)v(x)]_0^1 - \int_0^1 v\frac{du}{dx}dx = \left[\frac{1}{2}x^{2n}e^{x^2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}dx$

$= \frac{e}{2} - nI_{n-1}$.

Q5bii $I_0 = \int_0^1 xe^{x^2}dx = \left[\frac{1}{2}e^{x^2}\right]_0^1 = \frac{1}{2}(e-1)$.

$I_1 = \frac{e}{2} - I_0 = \frac{1}{2}$, $I_2 = \frac{e}{2} - 2I_1 = \frac{e}{2} - 1$.

Q5ci
$f(x) = \frac{e^x - e^{-x}}{2} - x$, $f'(x) = \frac{e^x + e^{-x}}{2} - 1$, $f''(x) = \frac{e^x - e^{-x}}{2}$.
For $x > 0$, $e^x > e^{-x}$, $\therefore f''(x) > 0$.

Q5cii
Since $f''(x) > 0$ for $x > 0$, $\therefore f'(x)$ is increasing for $x > 0$.

When $x = 0$, $f'(0) = \frac{e^0 + e^0}{2} - 1 = 0$.

$\therefore f'(x) > 0$ for $x > 0$.

Q5ciii
Since $f'(x) > 0$ for $x > 0$, $\therefore f(x)$ is increasing for $x > 0$.

When $x = 0$, $f(0) = \frac{e^0 - e^0}{2} - 0 = 0$.

$\therefore f(x) = \frac{e^x - e^{-x}}{2} - x > 0$ for $x > 0$.

Hence $\frac{e^x - e^{-x}}{2} > x$ for $x > 0$.

Q6a Translate the solid 4 units left, and reflect in the *y*-axis.
The base equation becomes $x = y^2$, i.e. $y = \pm\sqrt{x}$, $x \in [0,4]$.
Width of rectangle $= 2\sqrt{x}$, height of rectangle $= x$.

$V = \int_0^4 2\sqrt{x}.xdx = \int_0^4 2x^{\frac{3}{2}}dx = \left[\frac{4x^{\frac{5}{2}}}{5}\right]_0^4 = \frac{128}{5}$ cubic units.

Q6bi Let β be the remaining zero.

Product of zeros $= -1$, $\therefore -1 \times \alpha \times \beta = -1$, $\therefore \beta = \frac{1}{\alpha}$.

Q6bii(1) $P(x)$ has real coefficients, $\therefore \beta = \overline{\alpha}$.

$\therefore \overline{\alpha} = \frac{1}{\alpha}$, i.e. $\alpha\overline{\alpha} = 1$, $|\alpha|^2 = 1$. Hence $|\alpha| = 1$.

Q6bii(2) Sum of zeros $= -q$, $\therefore -1 + \alpha + \overline{\alpha} = -q$,
$\alpha + \overline{\alpha} = 1 - q$.

$\therefore 2\text{Re}(\alpha) = 1 - q$, $\text{Re}(\alpha) = \frac{1-q}{2}$.

Q6ci $PQ = \sqrt{OP^2 - OQ^2} = \sqrt{x^2 + y^2 - r^2}$.

Q6cii $PQ = PR$, i.e. $\sqrt{x^2 + y^2 - r^2} = c - x$,
$x^2 + y^2 - r^2 = (c - x)^2$.
Expand and simplify: $y^2 = r^2 + c^2 - 2cx$.

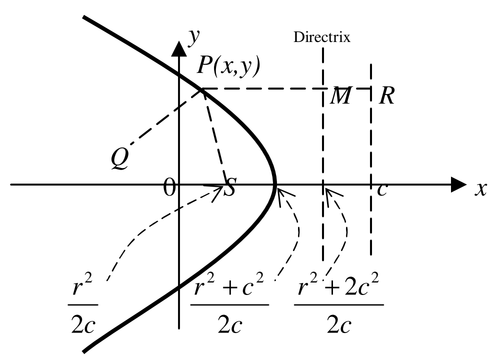
Q6ciii $y^2 = -2c\left(x - \frac{r^2 + c^2}{2c}\right)$ and compare with $Y^2 = -4aX$,

$a = \frac{c}{2}$. Distance between vertex and focus $= a = \frac{c}{2}$.

x-coordinate of $S = \frac{r^2 + c^2}{2c} - \frac{c}{2} = \frac{r^2}{2c}$. See diagram below.

$\therefore S$ is $\left(\frac{r^2}{2c}, 0\right)$.

Q6civ



$PS = PM = \frac{r^2 + 2c^2}{2c} - x$, $PQ = PR = c - x$.

$\therefore PS - PQ = \left(\frac{r^2 + 2c^2}{2c} - x\right) - (c - x) = \frac{r^2}{2c}$, independent of *x*.

Q7ai(1) $\ddot{x} = g - rv$, where $g - rv > 0$. Acceleration due to gravity is greater than that due to air resistance.

$v\frac{dv}{dx} = g - rv$, $\frac{dx}{dv} = \frac{v}{g-rv} = -\frac{1}{r}\left(\frac{g-rv-g}{g-rv}\right) = -\frac{1}{r}\left(1-\frac{g}{g-rv}\right)$.

$\therefore x = -\frac{1}{r}\int\left(1-\frac{g}{g-rv}\right)dv$, $rx = -v-\frac{g}{r}\ln(g-rv)+c$.

Given $x=0$ and $v=0$ initially, $0 = -\frac{g}{r}\ln(g)+c$,

$\therefore c = \frac{g}{r}\ln(g)$, $rx = \frac{g}{r}\ln\left(\frac{g}{g-rv}\right)-v$, $x = \frac{g}{r^2}\ln\left(\frac{g}{g-rv}\right)-\frac{v}{r}$.

Q7ai(2) Given $g = 9.8$, $r = 0.2$, and $v = 30$ when $x = L$,

$L = \frac{9.8}{0.2^2}\ln\left(\frac{9.8}{9.8-0.2\times 30}\right)-\frac{30}{0.2} \approx 82$ metres.

Q7aii When $x > L$, $x = e^{\frac{-t}{10}}(29\sin t - 10\cos t)+92$.

$v = \frac{dx}{dt} = -\frac{1}{10}e^{\frac{-t}{10}}(29\sin t-10\cos t)+e^{\frac{-t}{10}}(29\cos t+10\sin t)$

$= e^{\frac{-t}{10}}(-2.9\sin t+\cos t+29\cos t+10\sin t)$

$= e^{\frac{-t}{10}}(7.1\sin t+30\cos t)$.

Find x_{max} by letting $v = 0$.

Since $e^{\frac{-t}{10}} \neq 0$, $7.1\sin t + 30\cos t = 0$, $\tan t = -\frac{30}{7.1}$,

$t = \tan^{-1}\left(-\frac{30}{7.1}\right) \approx 1.8032$ and

$x_{max} = e^{\frac{-1.8032}{10}}(29\sin 1.8032-10\cos 1.8032)+92 \approx 117.49$.

Add the height of the jumper to x_{max}, 119.49 metres, which is less than 125 metres. The jumper's head stays out of the water.

Q7bi $z = cis\theta$, $z^n = cisn\theta = \cos n\theta + i\sin n\theta$,

$z^{-n} = cis(-n\theta) = \cos(-n\theta)+i\sin(-n\theta) = \cos n\theta - i\sin n\theta$.

$\therefore z^n + z^{-n} = 2\cos n\theta$.

Q7bii $(2\cos\theta)^{2m} = (z+z^{-1})^{2m}$

$= z^{2m}+\binom{2m}{1}z^{2m-1}z^{-1}+\binom{2m}{2}z^{2m-2}z^{-2}+\ldots+\binom{2m}{m-1}z^{m+1}z^{-m+1}$

$+\binom{2m}{m}z^m z^{-m}+\binom{2m}{m+1}z^{m-1}z^{-m-1}+\ldots+\binom{2m}{2m-1}z^1 z^{-2m+1}+z^{-2m}$

$= z^{2m}+\binom{2m}{1}z^{2m-2}+\binom{2m}{2}z^{2m-4}+\ldots+\binom{2m}{m-1}z^2$

$+\binom{2m}{m}+\binom{2m}{m+1}z^{-2}+\ldots+\binom{2m}{2m-1}z^{-2m+2}+z^{-2m}$

$= z^{2m}+z^{-2m}+\binom{2m}{1}z^{2m-2}+\binom{2m}{2m-1}z^{-2m+2}$

$+\binom{2m}{2}z^{2m-4}+\binom{2m}{2m-2}z^{-2m+4}+\ldots$

$+\binom{2m}{m-1}z^2+\binom{2m}{m+1}z^{-2}+\binom{2m}{m}$

Note: $\binom{2m}{1} = \binom{2m}{2m-1}$ etc.

$= (z^{2m}+z^{-2m})+\binom{2m}{1}(z^{2m-2}+z^{-2m+2})$

$+\binom{2m}{2}(z^{2m-4}+z^{-2m+4})+\ldots+\binom{2m}{m-1}(z^2+z^{-2})+\binom{2m}{m}$

$= 2\cos 2m\theta+\binom{2m}{1}2\cos(2m-2)\theta+\binom{2m}{2}2\cos(2m-4)\theta+\ldots$

$+\binom{2m}{m-1}2\cos 2\theta+\binom{2m}{m}$

$= 2\left[\cos 2m\theta+\binom{2m}{1}\cos(2m-2)\theta+\binom{2m}{2}\cos(2m-4)\theta+\ldots\right.$

$\left.+\binom{2m}{m-1}\cos 2\theta\right]+\binom{2m}{m}$

Q7biii $(2\cos\theta)^{2m} = 2^{2m}\cos^{2m}\theta$,

$\int_0^{\frac{\pi}{2}}\cos^{2m}\theta d\theta = \frac{1}{2^{2m}}\int_0^{\frac{\pi}{2}}(2\cos\theta)^{2m}d\theta$

$= \frac{2}{2^{2m}}\int_0^{\frac{\pi}{2}}\left[\cos 2m\theta+\binom{2m}{1}\cos(2m-2)\theta+\binom{2m}{2}\cos(2m-4)\theta+\ldots\right.$

$\left.+\binom{2m}{m-1}\cos 2\theta+\frac{1}{2}\binom{2m}{m}\right]d\theta$

$= \frac{2}{2^{2m}}\left[\frac{\sin 2m\theta}{2m}+\binom{2m}{1}\frac{\sin(2m-2)\theta}{2m-2}+\binom{2m}{2}\frac{\sin(2m-4)\theta}{2m-4}+\ldots\right.$

$\left.+\binom{2m}{m-1}\frac{\sin 2\theta}{2}+\frac{1}{2}\binom{2m}{m}\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{2^{2m}}\left[\frac{1}{2}\binom{2m}{m}\frac{\pi}{2}\right]$

Sine of 0 or multiple of π equals 0.

$= \frac{\pi}{2^{2m+1}}\binom{2m}{m}$

Q8ai $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}} = \frac{1-\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}} = \frac{1}{2\tan\frac{\theta}{2}}-\frac{\tan\frac{\theta}{2}}{2}$

$= \frac{1}{2}\cot\frac{\theta}{2}-\frac{1}{2}\tan\frac{\theta}{2}$.

$\therefore \cot\theta+\frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}$.

Q8aii From Q8ai, $\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}$.

$\therefore \tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x$.

$\therefore \sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$ is true for $n = 1$.

Assume it is true for $n = k$,

i.e. $\sum_{r=1}^{k} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x$.

When $n = k+1$,

$\sum_{r=1}^{k+1} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \sum_{r=1}^{k} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} + \frac{1}{2^{(k+1)-1}}\tan\frac{x}{2^{k+1}}$

$= \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x + \frac{1}{2^k}\tan\frac{x}{2^{k+1}}$

$= \frac{1}{2^{k-1}}\left(\cot\frac{x}{2^k} + \frac{1}{2}\tan\frac{1}{2}.\frac{x}{2^k}\right) - 2\cot x$

$= \frac{1}{2^{k-1}}\left(\frac{1}{2}\cot\frac{1}{2}.\frac{x}{2^k}\right) - 2\cot x$

$= \frac{1}{2^k}\cot\frac{x}{2^{k+1}} - 2\cot x$.

$\therefore \sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$ is true for $n = k+1$.

Hence, it is true for integers $n \geq 1$.

Q8aiii

$\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x = \frac{1}{2^{n-1}\tan\frac{x}{2^n}} - 2\cot x$,

As $n \to \infty$, $\tan\frac{x}{2^n} \to \frac{x}{2^n}$.

Hence, $\lim_{n\to\infty}\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} \to \frac{1}{2^{n-1}.\frac{x}{2^n}} - 2\cot x = \frac{2}{x} - 2\cot x$.

Q8aiv Let $x = \frac{\pi}{2}$, $\tan\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{8} + \frac{1}{4}\tan\frac{\pi}{16} + ...$

$= \lim_{n\to\infty}\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{\frac{\pi}{2}}{2^r}$

$= \frac{2}{\frac{\pi}{2}} - 2\cot\frac{\pi}{2} = \frac{4}{\pi}$.

Q8b $A_{small} < \int_{n-1}^{n} \frac{1}{x}dx < A_{l\arg e}$, $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$,

$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$, $e^{-\frac{1}{n}} > \frac{n-1}{n} > e^{-\frac{1}{n-1}}$, $e^{-\frac{1}{n-1}} < \frac{n-1}{n} < e^{-\frac{1}{n}}$,

$\left(e^{-\frac{1}{n-1}}\right)^n < \left(1 - \frac{1}{n}\right)^n < \left(e^{-\frac{1}{n}}\right)^n$, $\therefore e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$.

Q8ci Let A_1 represent A_1 wins, and $A_1 i$ represent A_1 wins in the ith draw.

For any one, in a single draw, probability of winning $= p = \frac{1}{n}$,

and probability of not winning $= q = 1 - \frac{1}{n}$.

$\Pr(A_1) = \Pr(A_1 1) + \Pr(A_1 2) + \Pr(A_1 3) + ...$

$\therefore W = p + q^n p + q^{2n} p + q^{3n} p + ...$

$\therefore W = p + q^n\left(p + q^n p + q^{2n} p + q^{3n} p + ...\right)$

$\therefore W = p + q^n W$.

Q8cii $W_m = p + q^n p + q^{2n} p + ... + q^{(m-1)n} p = \frac{p\left(1-q^{mn}\right)}{1-q^n}$.

[Sum of a GP with $a = p$ and $r = q^n$]

From Q8ci, $W = p + q^n W$, $\therefore W = \frac{p}{1-q^n}$.

$\therefore \frac{W_m}{W} = 1 - q^{mn} = 1 - \left(1 - \frac{1}{n}\right)^{mn}$.

From Q8b, $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$, and $e^{-\frac{n}{n-1}} \to e^{-1}$ for large n.

$\therefore \left(1-\frac{1}{n}\right)^n \approx e^{-1}$ for large n.

$\therefore \frac{W_m}{W} = 1 - \left(1-\frac{1}{n}\right)^{mn} \approx 1 - \left(e^{-1}\right)^m = 1 - e^{-m}$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.

