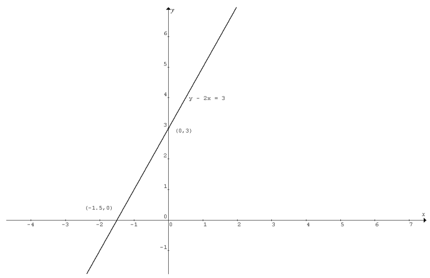
2009 NSW BOS Mathematics Exam Solutions

Q1a



Q1b $\frac{5x-4}{x}=2$, $5x-4=2x$, $3x=4$, $x=\frac{4}{3}$.

Q1c $|x+1|=5$, $x+1=\pm 5$, $x=-6$ or 4.

Q1d $y=x^4-3x$, $\frac{dy}{dx}=4x^3-3$.
At $(1,-2)$, gradient of tangent $=4\times 1^3-3=1$.

Q1e $2\cos\theta=1$, $\theta=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

Q1f $\ln x=2$, $x=e^2=7.3891$.

Q2ai $\frac{d}{dx}(x\sin x)=\sin x+x\cos x$ by the product rule.

Q2aii $\frac{d}{dx}\left(e^x+1\right)^2=2e^x\left(e^x+1\right)$ by the chain rule.

Q2bi $\int 5dx=5x+c$

Q2bii $\int\frac{3}{(x-6)^2}dx=\int 3(x-6)^{-2}dx=\frac{3(x-6)^{-1}}{-1}+c=\frac{-3}{x-6}+c$.

Q2biii $\int_1^4\left(x^2+\sqrt{x}\right)dx=\left[\frac{x^3}{3}+\frac{2x^{\frac{3}{2}}}{3}\right]_1^4=\frac{80}{3}-1=\frac{77}{3}$.

Q2c $\sum_{k=1}^{4}(-1)^k k^2=(-1)^1 1^2+(-1)^2 2^2+(-1)^3 3^2+(-1)^4 4^2$
$=-1+4-9+16=10$.

Q3a Use $S_n=\frac{n}{2}(a+\ell)$, $S_{21}=\frac{21}{2}(3+53)=588$.

Q3bi Gradient of tangent $LM=\frac{5-1}{5-2}=\frac{4}{3}$.

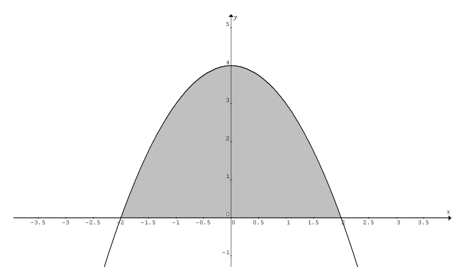
Equation of tangent LM: $y-1=\frac{4}{3}(x-2)$, $4x-3y-5=0$.

Q3bii Equation of radius NP: $y-3=-\frac{3}{4}(x-1)$,
$3x+4y-15=0$.
Solve equations LM and NP simultaneously to find intersection P: $P\left(\frac{13}{5},\frac{9}{5}\right)$.

Distance $NP=\sqrt{\left(1-\frac{13}{5}\right)^2+\left(3-\frac{9}{5}\right)^2}=2$.

Q3biii $(x-1)^2+(y-3)^2=4$.

Q3c



Q3d
$Area\approx\frac{50}{3}(210+4\times 220+2\times 200+4\times 190+2\times 210+4\times 240+240)$
$=64500\text{ m}^2$.

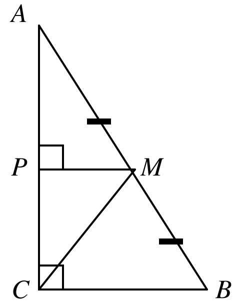
Q4a Limiting height is $S_\infty=\frac{a}{1-r}=\frac{1.2}{1-\frac{9}{10}}=12$ m.

Q4b $x^2-(k+4)x+(k+7)=0$.
Equal roots, $\Delta=0$, $[-(k+4)]^2-4\times 1\times(k+7)=0$,
$k^2+4k-12=0$, $(k+6)(k-2)=0$, $k=-6$ or 2.

Q4ci For ΔAPM and ΔACB ,
$\angle APM=\angle ACB=90°$,
$\angle PAM=\angle CAB$ (common angle),
$\therefore\angle AMP=\angle ABC$.
All corresponding angles are equal, hence the two triangles are similar.

Q4cii $AP : AC = AM : AB = 1:2$.

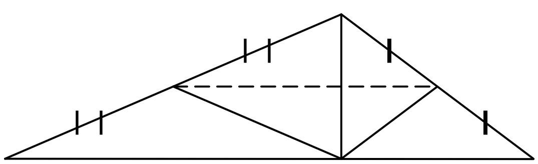
Q4ciii



Since $AP : AC = 1:2$, $\therefore AP = PC$.
PM is a common side for ΔAPM and ΔCPM.
$\angle APM = \angle CPM = 90°$.
$\therefore \Delta APM$ and ΔCPM are congruent.
$\therefore \angle PAM = \angle PCM$. Hence ΔAMC is isosceles.

Q4civ Since ΔAMC is isosceles, $\therefore MB = AM = MC$.
Hence ΔMCB is isosceles as well.
$\therefore \Delta ABC$ can be divided into two isosceles triangles.

Q4cv



Q5ai Line AB: $y = \sqrt{3}x - 3$. When $y = 0$, $x = \sqrt{3}$. $\therefore B\left(\sqrt{3}, 0\right)$.

Gradient of line $BC = -\frac{1}{\sqrt{3}}$, and equation of BC:

$y = -\frac{1}{\sqrt{3}}\left(x - \sqrt{3}\right)$, $\therefore y = -\frac{1}{\sqrt{3}}x + 1$.

Q5aii $A(0,-3)$ and $C(0,1)$, $\therefore CA = 1 - (-3) = 4$.

$Area\Delta ABC = \frac{1}{2} \times CA \times OB = \frac{1}{2} \times 4 \times \sqrt{3} = 2\sqrt{3}$ square units.

Q5bi $\frac{1}{3}$

Q5bii $\Pr(all3levels) = \frac{2}{3} \times \frac{1}{2} \times 1 = \frac{1}{3}$.

Q5biii Binomial: $n = 5$, $p = \frac{2}{3}$.

$\Pr(X = 5) = \left(\frac{2}{3}\right)^5 = \frac{32}{243}$.

Q5ci $A = \frac{1}{2} r^2 \sin\theta$, where $0 \le \theta \le \pi$.

$\sqrt{3} = \frac{1}{2} \times 2^2 \sin\theta$, $\sin\theta = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

The other value of θ is $\frac{2\pi}{3}$.

Q5cii(1) Area of sector $AOB = \frac{\frac{\pi}{3}}{2\pi} \times \pi r^2 = \frac{2\pi}{3}$ cm^2.

Q5cii(2) Chord $AB = \sqrt{2^2 + 2^2 - 2 \times 2 \times 2\cos\frac{\pi}{3}} = 2$ cm.

Arc $AB = r\theta = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$ cm.

Perimeter of minor segment $= 2 + \frac{2\pi}{3}$ cm.

Q6a $V = 2 \times \int_0^{\frac{\pi}{3}} \pi y^2 dx = 2\pi \int_0^{\frac{\pi}{3}} \sec^2 x dx = 2\pi\left[\tan x\right]_0^{\frac{\pi}{3}} = 2\sqrt{3}\pi$ cubic units.

Q6bi $Q = Ae^{-kt}$, $k > 0$. When $t = 1600$, $Q = \frac{A}{2}$,

$\therefore \frac{A}{2} = Ae^{-k1600}$, $e^{1600k} = 2$, $k = \frac{1}{1600}\log_e 2 \approx 0.000433217$.

Q6bii Safe level, $Q = \frac{A}{3}$. $Ae^{-0.000433217t} = \frac{A}{3}$, $\therefore e^{0.000433217t} = 3$.

Hence $t = \frac{\ln 3}{0.000433217} \approx 2536$ years.

Q6ci $y = ax^2 + bx$, $\frac{dy}{dx} = 2ax + b$.

At $O(0,0)$, $\frac{dy}{dx} = 2a0 + b = 1.2$, $\therefore b = 1.2$.

At $P(30,...)$, $\frac{dy}{dx} = 2a30 + 1.2 = -1.8$, $\therefore a = -0.05$.

Q6cii $\therefore y = -0.05x^2 + 1.2x$.
When $x = 30$, $y = -9$. $\therefore P(30,-9)$.

Turning point: $\frac{dy}{dx} = 2ax + b = 2(-0.05)x + 1.2 = 0$, $\therefore x = 12$ and

$y = -0.05 \times 12^2 + 1.2 \times 12 = 7.2$.
$\therefore d = 7.2 - (-9) = 16.2$ m.

Q7ai $\ddot{x} = 8e^{-2t} + 3e^{-t}$, $x = 5$ and $\dot{x} = -6$ when $t = 0$.
$\therefore \dot{x} = -4e^{-2t} - 3e^{-t} + 1$ and $x = 2e^{-2t} + 3e^{-t} + t$.

Q7aii The particle comes to rest: $\dot{x} = -4e^{-2t} - 3e^{-t} + 1 = 0$
$\left(-4e^{-t} + 1\right)\left(e^{-t} + 1\right) = 0$.

Since $e^{-t} + 1 > 0$, $\therefore -4e^{-t} + 1 = 0$, $e^{-t} = \frac{1}{4}$, $e^{t} = 4$, $t = \ln 4$.

Q7aiii The particle comes to rest when $e^{-t} = \frac{1}{4}$,

$\therefore x = 2e^{-2t} + 3e^{-t} + t = 2\left(\frac{1}{4}\right)^2 + 3\times\frac{1}{4} + \ln 4 = \frac{7}{8} + \ln 4$.

Q7bi $h = 1 + 0.7\sin\frac{\pi}{6}t$, $0 \le t \le 12$, i.e. from 5 am to 5 pm.

Period $= \frac{2\pi}{\frac{\pi}{6}} = 12$ hours.

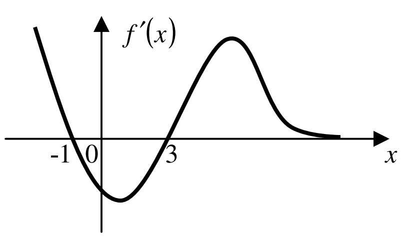
Q7bii At low tide, $\sin\frac{\pi}{6}t = -1$, $\frac{\pi}{6}t = \frac{3\pi}{2}$, $t = 9$, i.e. 2 pm, and $h = 1 - 0.7 = 0.3$ m .

Q7biii Let $1 + 0.7\sin\frac{\pi}{6}t = 1.35$, $\sin\frac{\pi}{6}t = \frac{1}{2}$, $\frac{\pi}{6}t = \frac{\pi}{6}$, $\frac{5\pi}{6}$
$\therefore t = 1$, 5. The ship was able to enter the harbour from 6 am to 10 am.

Q8ai $-1 < x < 3$

Q8aii As $x \to \infty$, $f'(x) \to 0^+$.

Q8aiii $f'(x) = 0$ when $x = -1$, 3 or $x \to \infty$.



Q8bi After one month and before repayment, amount owed $= 350000\times\left(1 + \frac{\frac{9}{12}}{100}\right) = 352625$.

After repayment, amount owed $= 352625 - 2937 = \$349688$.

Q8bii $A_n = 0$ when $n = 288$.
$\therefore 346095\times1.005^{288} - \left(1 + 1.005 + 1.005^2 + ... + 1.005^{287}\right)M = 0$,

$346095\times1.005^{288} - \frac{1\left(1.005^{288} - 1\right)}{1.005 - 1}M = 0$,

$M = \$2270.31$

Q8biii $346095\times1.005^{n} - \frac{1\left(1.005^{n} - 1\right)}{1.005 - 1}\times2937 = 0$,

$346095\times1.005^{n} - \frac{2937\left(1.005^{n} - 1\right)}{0.005} = 0$,

$346095\times1.005^{n} - 587400\left(1.005^{n} - 1\right) = 0$,

$1.005^{n} = 2.4342637$, $n = \frac{\ln 2.4342637}{\ln 1.005} \approx 178.37332$ months,

$\therefore$ it will take 14 years 11 months.

Q8biv Save $2270.31\times288 - 2937\times178.37332 = \$129967$ to the nearest dollar.

Q9a $\Pr(at\_least\_one) = 1 - \Pr(none)$

$= 1 - \left(1 - \frac{1}{9}\right)^3\left(1 - \frac{1}{16}\right)^3 = \frac{91}{216}$.

Q9bi From P to R and then from R to S:
Total cost $= 1000\times5 + 2600\times3 = \$12800$.

Q9bii From P to S: $= 2600\times\sqrt{5^2 + 3^2} = \$15160.47$.

Q9biii $C = 1000(5 - x) + 2600\sqrt{x^2 + 3^2}$

$= 1000\left(5 - x + 2.6\sqrt{x^2 + 9}\right)$, where $0 \le x \le 5$.

Q9biv $\frac{dC}{dx} = 1000\left(-1 + \frac{2.6x}{\sqrt{x^2 + 9}}\right) = 0$,

$\therefore \frac{2.6x}{\sqrt{x^2 + 9}} = 1$, $2.6x = \sqrt{x^2 + 9}$, $5.76x^2 = 9$, $x = 1.25$.

Minimum cost $= 1000\left(5 - 1.25 + 2.6\sqrt{1.25^2 + 9}\right) = \$12200$.

Q9bv $C = 1000(5 - x) + 1100\sqrt{x^2 + 3^2}$

$= 1000\left(5 - x + 1.1\sqrt{x^2 + 9}\right)$, where $0 \le x \le 5$.

C is a decreasing function in $0 \le x \le 5$, minimum cost is achieved when $x = 5$, and the path is from P to S in a straight line.

Q10a $f(x)=x-\frac{x^2}{2}+\frac{x^3}{3}$, $f'(x)=1-x+x^2$.

$\Delta=(-1)^2-4(1)(1)=-3<0$ for $f'(x)$, $\therefore f'(x)\neq 0$. Hence, $f(x)$ has no turning points.

Q10b At the point of inflexion, $f''(x)=-1+2x=0$, $x=\frac{1}{2}$ and $y=f\left(\frac{1}{2}\right)=\frac{5}{12}$. Point of inflexion is $\left(\frac{1}{2},\frac{5}{12}\right)$.

Q10ci $L.H.S.=1-x+x^2-\frac{1}{1+x}=\frac{(1-x+x^2)(1+x)-1}{1+x}$

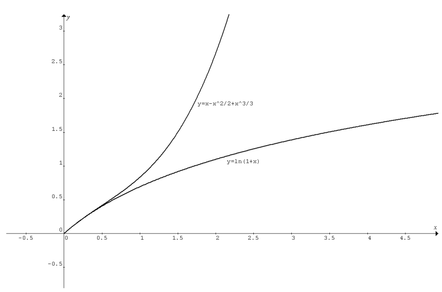
$=\frac{1+x^3-1}{1+x}=\frac{x^3}{1+x}=R.H.S.$

Q10cii $g(x)=\ln(1+x)$, $g'(x)=\frac{1}{1+x}$.

$f'(x)-g'(x)=1-x+x^2-\frac{1}{1+x}=\frac{x^3}{1+x}$.

Hence, for $x\geq 0$, $f'(x)-g'(x)\geq 0$, $\therefore f'(x)\geq g'(x)$.

Q10d



Q10e $\frac{d}{dx}[(1+x)\ln(1+x)-(1+x)]=\frac{d}{dx}[(1+x)(\ln(1+x)-1)]$

$=1(\ln(1+x)-1)+(1+x)\times\frac{1}{(1+x)}=\ln(1+x)$.

Q10f Enclosed area $=\int_0^1\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ln(1+x)\right)dx$

$=\left[\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{12}-((1+x)\ln(1+x)-(1+x))\right]_0^1$

$=\left(\frac{29}{12}-2\ln 2\right)-(1)=\frac{17}{12}-\ln 4$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.