2010 NSW BOS Mathematics Extension 1 Solutions

Q1a $\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\left(\frac{x}{2}\right)+c$

Q1b $f(x)=\cos^{-1}\left(\frac{x}{2}\right)$, $-1\le \frac{x}{2}\le 1$, $-2\le x\le 2$.

Domain is $[-2,2]$.

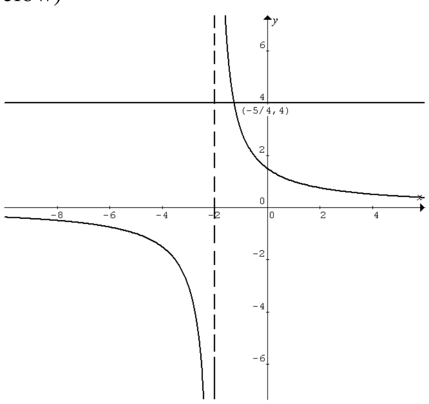
Q1c $\ln(x+6)=2\ln x$, $\therefore$ $x+6>0$ and $x>0$, $\therefore$ $x>0$

$x+6=x^2$, $x^2-x-6=0$, $(x-3)(x+2)=0$

$\therefore$ $x=3$

Q1d Let $\frac{3}{x+2}=4$, $\therefore$ $x=-\frac{5}{4}$

For $\frac{3}{x+2}<4$, $x<-2$ or $x>-\frac{5}{4}$ (see the graphs of $y=\frac{3}{x+2}$ and $y=4$ below)



Q1e Let $u=1-x$, $-\frac{du}{dx}=1$, $x=1-u$

$$\int_0^1 x\sqrt{1-x}\,dx = -\int_0^1 (1-u)\sqrt{u}\frac{du}{dx}dx = \int_0^1\left(u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du$$

$$=\left[\frac{2u^{\frac{3}{2}}}{3}-\frac{2u^{\frac{5}{2}}}{5}\right]_0^1=\frac{4}{15}$$

Q1f Binomial: $n=5$, $p=\frac{1}{6}$, $q=\frac{5}{6}$

$$\Pr(X=2)={}^5C_3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

Q2a $f'(x)=\sin^2 x=\frac{1}{2}(1-\cos 2x)$

$\therefore$ $f(x)=\frac{1}{2}\int(1-\cos 2x)dx=\frac{1}{2}x-\frac{1}{4}\sin 2x+c$

Given $f(0)=2$, $\therefore$ $f(x)=\frac{1}{2}x-\frac{1}{4}\sin 2x+2$

Q2bi $M=36-35.5e^{-kt}$, $\therefore$ $35.5e^{-kt}=36-M$

$\therefore$ $\frac{dM}{dt}=35.5ke^{-kt}=k(36-M)$

Q2bii Given $M=20$ when $t=10$, $\therefore$ $20=36-35.5e^{-10k}$

$\therefore$ $e^{-10k}=\frac{16}{35.5}=0.450704$, $k=\frac{\ln 0.450704}{-10}=0.080$

Q2biii Let $\frac{dM}{dt}=k(36-M)=0$, limiting $M=36$ tonnes

Q2ci $P(x)=(x+1)(x-3)Q(x)+ax+b$

$P(x)$ has a factor of $x-3$, $\therefore$ $ax+b=a(x-3)$, $\therefore$ $b=-3a$

When $P(x)$ is divided by $x+1$, the remainder is

$P(-1)=-a+b=8$, $\therefore$ $-4a=8$, $\therefore$ $a=-2$ and $b=6$

Q2cii When $P(x)=(x+1)(x-3)Q(x)-2x+6$ is divided by $(x+1)(x-3)$, the remainder is $-2x+6$.

Q2d $r=\sqrt{x^2+36}$ and $\frac{dx}{dt}=100$

$$\frac{dr}{dt}=\frac{x}{\sqrt{x^2+36}}\times\frac{dx}{dt}=\frac{100x}{\sqrt{x^2+36}}$$

Q3ai $\frac{5!}{2!}=60$

Q3aii $4!=24$

Q3bi $f(x)=e^{-x^2}$, $f'(x)=-2xe^{-x^2}$

$f''(x)=(-2x)^2e^{-x^2}+(-2)e^{-x^2}$

Let $f''(x)=0$, $(-2x)^2e^{-x^2}+(-2)e^{-x^2}=0$

$\therefore$ $2(2x^2-1)e^{-x^2}=0$, $2x^2-1=0$, $\therefore$ $x=\pm\frac{1}{\sqrt{2}}$ are the *x*-coordinates of the points of inflexion.

Q3bii $f(x)$ is a many-to-one function, it has an inverse but not an inverse function. By restricting the domain $f(x)$ can be made a one-to-one function and has an inverse function.

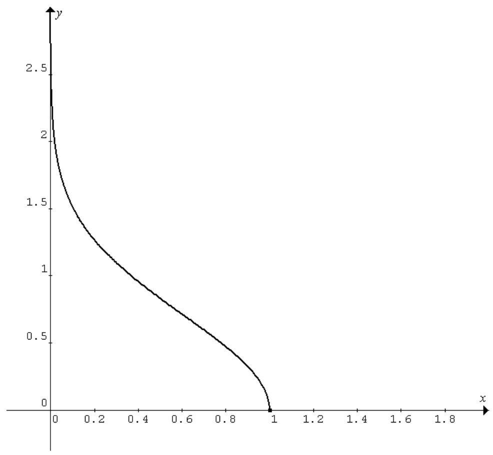
Q3biii Let $y=f(x)=e^{-x^2}$, $x\ge 0$, $\therefore$ $y\in(0,1]$

Equation of inverse: $x=e^{-y^2}$, $y^2=-\ln x=\ln\frac{1}{x}$, $\therefore$ $y=\sqrt{\ln\frac{1}{x}}$

$\therefore$ $f^{-1}(x)=\sqrt{\ln\frac{1}{x}}$

Q3biv The domain of $f^{-1}(x)$ is the range of the restricted $f(x)$, i.e. $(0,1]$

Q3bv



Q3bvi(1) At $x = 0.6$, $x < e^{-x^2}$; at $x = 0.7$, $x > e^{-x^2}$

.: $y = x$ and $y = e^{-x^2}$ intersect at $0.6 < x < 0.7$.

Hence $x = e^{-x^2}$ has a solution in the interval $0.6 < x < 0.7$.

Q3bvi(2) At $x = 0.65$, $x < e^{-x^2}$, .: the solution to $x = e^{-x^2}$ is > 0.65, .: $x = 0.7$ correct to one decimal place.

Q4ai $v^2 = 24 - 8x - 2x^2$

When $v = 0$, $24 - 8x - 2x^2 = 0$, $2(6 + x)(2 - x) = 0$

.: $x = -6$ or 2

Q4aii $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4 - 2x$

Q4aiii Let $a = 0$, $x = -2$, .: $v^2 = 32$, $v = \pm\sqrt{32} = \pm 4\sqrt{2}$

Maximum speed $= |v| = 4\sqrt{2}$

Q4bi

$R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right) = 2\cos\theta + 2\cos\theta\cos\frac{\pi}{3} - 2\sin\theta\sin\frac{\pi}{3}$

$= 3\cos\theta - \sqrt{3}\sin\theta$

.: $R\cos\alpha = 3$ and $R\sin\alpha = \sqrt{3}$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 9 + 3$, .: $R = 2\sqrt{3}$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3}$, .: $\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$

Hence $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right)$

Q4bii $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right) = 3$ and $0 < \theta < 2\pi$

$2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) = 3$, .: $\cos\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

.: $\theta + \frac{\pi}{6} = \frac{11\pi}{6}$, $\theta = \frac{5\pi}{3}$

Q4c Equation of the tangent at $P(2ap, ap^2)$:

Gradient $= \frac{\frac{dy}{dp}}{\frac{dx}{dp}} = \frac{2ap}{2a} = p$, .: $y = px + c$, .: $ap^2 = p(2ap) + c$

.: $c = -ap^2$, $y = px - ap^2$

.: y-intercept of the tangent is $L(0, -ap^2)$

Point $M(2ap, -a)$

Length $SL = a + ap^2$, length $PM = ap^2 + a$,

length $PS = \sqrt{(2ap)^2 + (ap^2 - a)^2} = ap^2 + a$,

length $ML = \sqrt{(2ap)^2 + (-ap^2 + a)^2} = ap^2 + a$

All sides of *SLMP* are equal, .: it is a rhombus.

Q5ai $\frac{1}{AP} = \tan 20°$, $AP = \frac{1}{\tan 20°}$

$PT = AP\tan 3° = \frac{\tan 3°}{\tan 20°}$

$\frac{PT}{BP} = \tan 30° = \frac{1}{\sqrt{3}}$, .: $BP = \sqrt{3}PT = \frac{\sqrt{3}\tan 3°}{\tan 20°}$

Q5aii $AB = AP - BP = \frac{1}{\tan 20°} - \frac{\sqrt{3}\tan 3°}{\tan 20°} = \frac{1 - \sqrt{3}\tan 3°}{\tan 20°}$

≈ 2.498 km

Q5bi $f(x) = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1 + x^2} + \frac{-\frac{1}{x^2}}{1 + \left(\frac{1}{x}\right)^2} = 0$, .: $f(x)$ is a constant for any $x > 0$

Let $x = 1$, $f(1) = \tan^{-1}1 + \tan^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$, .: $f(x) = \frac{\pi}{2}$

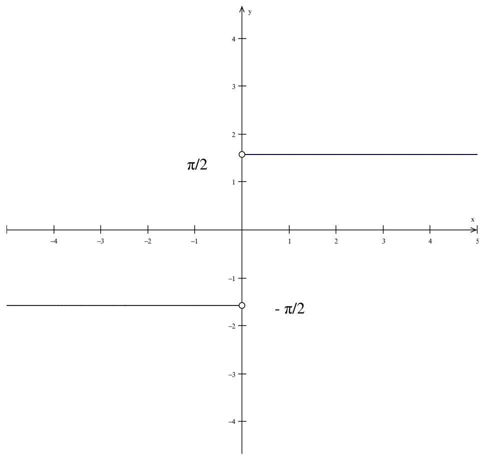
Note that $0 < f(x) < \pi$ for $x > 0$, and

$\tan(f(x)) = \tan\left(\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)\right)$

$= \frac{\tan(\tan^{-1}x) + \tan\left(\tan^{-1}\left(\frac{1}{x}\right)\right)}{1 - \tan(\tan^{-1}x)\tan\left(\tan^{-1}\left(\frac{1}{x}\right)\right)} = \frac{x + \frac{1}{x}}{1 - x \times \frac{1}{x}}$, which is undefined

.: $f(x) = \frac{\pi}{2}$, the only value in $(0, \pi)$ that tan is undefined.

Q5bii



Q5ci $\angle ABD + \angle XDB + \angle BXD = 180°$
.: $\angle ABD + \angle XDB = 180° - \angle BXD = \angle AXD$

Q5cii $\angle AXD = \angle TAD$ (angle between a tangent and a chord equals to angle in the alternate segment)
.: $\angle AXD = \angle TAC + \angle CAD$

Q5ciii From i and ii, $\angle ABD + \angle XDB = \angle TAC + \angle CAD$
$\angle ABD = \angle ABC = \angle TAC$ (alternate segment theorem)
.: $\angle XDB = \angle CAD$
$\angle XDB = \angle DAB$ (alternate segment theorem)
.: $\angle CAD = \angle DAB$
Hence AD bisects $\angle BAC$.

Q6ai $\cos(A-B) = \cos A \cos B + \sin A \sin B$
$= \cos A \cos B\left(1 + \frac{\sin A \sin B}{\cos A \cos B}\right)$
$= \cos A \cos B(1 + \tan A \tan B)$

Q6aii $B < A < \pi$.: $0 < A - B < \pi - B$
$0 < B < \frac{\pi}{2}$, .: $0 < A - B < \frac{\pi}{2}^{+}$
If $\tan A \tan B = -1$, $\cos(A-B) = 0$, $A - B = \frac{\pi}{2}$

Q6bi $x = vt\cos\theta$, $t = \frac{x}{v\cos\theta}$,
$y = vt\sin\theta - 5t^2$
.: $y = \frac{x\sin\theta}{\cos\theta} - 5\left(\frac{x}{v\cos\theta}\right)^2 = \frac{x\sin\theta}{\cos\theta} - 5\left(\frac{x}{v\cos\theta}\right)^2$
.: $h = \frac{d\sin\theta}{\cos\theta} - 5\left(\frac{d}{v\cos\theta}\right)^2$, $\frac{h}{d} = \frac{\sin\theta}{\cos\theta} - \frac{5d}{v^2\cos^2\theta} = \tan\alpha$
.: $v^2(\cos\theta\sin\theta - \cos^2\theta\tan\alpha) = 5d$
.: $v^2 = \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha}$

Q6bii(1) As $\theta \to \alpha$, $\cos\theta\sin\theta - \cos^2\theta\tan\alpha \to 0$
$v^2 = \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha} \to \infty$, .: $v \to \infty$

Q6bii(2)
$v^2 = \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha} = \frac{5d}{\cos\theta(\sin\theta - \cos\theta\tan\alpha)}$
As $\theta \to \frac{\pi}{2}$, $\sin\theta - \cos\theta\tan\alpha \to 1$, $v^2 \to \frac{5d}{\cos\theta} \to \infty$.: $v \to \infty$

Q6biii
$F(\theta) = \cos\theta\sin\theta - \cos^2\theta\tan\alpha = \cos\theta(\sin\theta - \cos\theta\tan\alpha)$
$F'(\theta) = \cos\theta(\cos\theta + \sin\theta\tan\alpha) - \sin\theta(\sin\theta - \cos\theta\tan\alpha)$
$= \cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta\tan\alpha$
$= \cos 2\theta + \sin 2\theta \tan\alpha$
$= \cos 2\theta + \cos 2\theta \tan 2\theta \tan\alpha = \cos 2\theta - \cos 2\theta = 0$

Q6biv From part aii, if $\tan 2\theta \tan\alpha = -1$,
then $2\theta - \alpha = \frac{\pi}{2}$, i.e. $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$
.: $F'(\theta) = 0$ when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$

Q6bv From part bii, $v^2 \to \infty$ as $\theta \to \alpha$ or $\theta \to \frac{\pi}{2}$, .: v^2 is a minimum when $0 < \alpha < \theta < \frac{\pi}{2}$.
From part biv, $F'(\theta) = 0$ when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$ only.
.: $F(\theta)$ is a maximum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$ because v^2 is a minimum.
.: v^2 is a minimum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$.

Q7a Let $I_n = 47^n + 53\times 147^{n-1}$
$I_1 = 47 + 53 = 100$ is divisible by 100
Assume $I_k = 47^k + 53\times 147^{k-1}$ is divisible by 100
$I_{k+1} = 47^{k+1} + 53\times 147^k = 47\times 47^k + 53\times 147\times 147^{k-1}$
$= 47\times 47^k + 47\times 53\times 147^{k-1} + 100\times 53\times 147^{k-1}$
$= 47(47^k + 53\times 147^{k-1}) + 100\times 53\times 147^{k-1}$
Both terms in I_{k+1} are divisible by 100, .: I_{k+1} is divisible by 100.
Hence $I_n = 47^n + 53\times 147^{n-1}$ is divisible by 100 for all integers $n \ge 1$.

Q7bi $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n$

Let $x = 1$, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n} = \sum_{k=0}^{n}\binom{n}{k}$

Q7bii Let $n = 100$,

$\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + ... + \binom{100}{100} = 2^{100}$

Q7biii

$$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n\right)$$

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + ... + n\binom{n}{n}x^{n-1}$$

Let $x = 1$, $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = \sum_{k=1}^{n} k\binom{n}{k}$

Q7ci The selection may have 0 red balls, 1 red ball, 2, 3, ..., r red balls. .: $r+1$ possible combinations.

Q7cii Selecting $n-r$ from n different balls, number of ways $= \binom{n}{n-r} = \binom{n}{r}$

Q7ciii Using the results of parts ci and cii, there are $r+1$ ways to select r red/blue balls and $\binom{n}{r}$ ways to select $n-r$ white balls.

.: number of ways to select n balls consisting r red/blue balls and $n-r$ white balls is $(r+1)\binom{n}{r}$.

.: number of different selections

$$= \sum_{r=0}^{n}(r+1)\binom{n}{r} = \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r} = \sum_{r=1}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r}$$

$$= n2^{n-1} + 2^n = n2^{n-1} + 2 \times 2^{n-1}$$

$$= (n+2)2^{n-1}$$

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