2010 NSW BOS Mathematics Extension 2 Solutions

Q1a Let $u = 1 + 3x^2$, $\frac{1}{6}\frac{du}{dx} = x$

$\int \frac{x}{\sqrt{1+3x^2}}dx = \int \frac{1}{6u^{\frac{1}{2}}}\frac{du}{dx}dx = \int \frac{1}{6u^{\frac{1}{2}}}du$

$= \frac{1}{3}u^{\frac{1}{2}} + c = \frac{1}{3}\sqrt{1+3x^2} + c$

Q1b Let $u = \cos x$, $-\frac{du}{dx} = \sin x$

$\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}dx = \int_0^{\frac{\pi}{4}} -\frac{1}{u}\frac{du}{dx}dx = \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{u}du$

$\left[-\ln u\right]_1^{\frac{1}{\sqrt{2}}} = \ln\sqrt{2} = \frac{1}{2}\ln 2$

Q1c $\int \frac{1}{x(x^2+1)}dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$ [Partial fractions]

$= \ln|x| - \frac{1}{2}\ln(x^2+1) + c = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$

Q1d $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$, $2\frac{dt}{dx} = 1 + t^2$

$\tan x = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$,

$1 + \sin x = 1 + \frac{2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = \int_0^{\frac{\pi}{2}} \frac{(1+t^2)dx}{(1+t)^2} = \int_0^{\frac{\pi}{2}} \frac{2}{(1+t)^2}\frac{dt}{dx}dx$

$= \int_0^1 \frac{2dt}{(1+t)^2} = \left[-\frac{2}{1+t}\right]_0^1 = 1$

Q1e Let $u = \sqrt{x}$, $u^2 = x$, $\frac{dx}{du} = 2u$, $2u\frac{du}{dx} = 1$

$\int \frac{1}{1+\sqrt{x}}dx = \int \frac{1}{1+u}2u\frac{du}{dx}dx = \int \frac{1}{1+u}2udu = 2\int \frac{u}{1+u}du$

$= 2\int\left(1 - \frac{1}{1+u}\right)du = 2u - 2\ln(1+u) + c = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + c$

Q2ai $z^2 = (5-i)^2 = 25 - 10i + i^2 = 24 - 10i$

Q2aii $z + 2\bar{z} = 5 - i + 2(5+i) = 15 + i$

Q2aiii $\frac{i}{z} = \frac{i\bar{z}}{z\bar{z}} = \frac{i(5+i)}{26} = -\frac{1}{26} + \frac{5}{26}i$

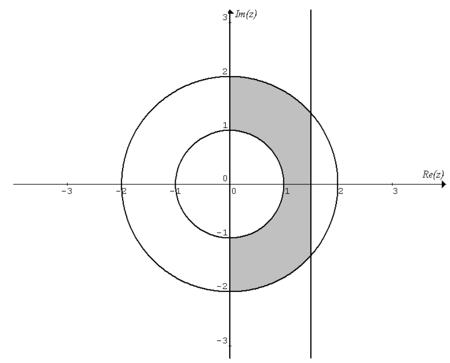
Q2bi $-\sqrt{3} - i$ is in the third quadrant.

$r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$, $\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\frac{5\pi}{6}$

$-\sqrt{3} - i = 2cis\left(-\frac{5\pi}{6}\right)$

Q2bii $\left(-\sqrt{3} - i\right)^6 = 2^6 cis\left(-\frac{5\pi}{6}\times 6\right) = 64cis\pi = -64 \in R$

Q2c $1 \le |z| \le 2$ is the region between two concentric circles of radii 1 and 2 and centred at O.
$0 \le z + \bar{z} \le 3$, $0 \le 2x \le 3$, $0 \le x \le 1.5$ is the region between vertical lines $x = 0$ and $x = 1.5$.



Q2di $\overline{OA} = |z| = |\cos\theta + i\sin\theta| = 1$

$\overline{OB} = |z^2| = |\cos 2\theta + i\sin 2\theta| = 1$

Since $OACB$ is a parallelogram, .: $\overline{BC} = \overline{OA} = 1$ and $\overline{AC} = \overline{OB} = 1$.
.: $OACB$ is a rhombus.

Q2dii $\angle AOB = \arg z^2 - \arg z = 2\theta - \theta = \theta$

$OACB$ is a rhombus, .: OC bisects $\angle AOB$, .: $\angle COA = \frac{\theta}{2}$

.: $\arg\left(z + z^2\right) = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$

Q2diii Let AB intersects OC at D. $OACB$ is a rhombus, .: AD is a perpendicular bisector of OC.
.: ΔOAD is a right-angled triangle.
.: $\overline{OD} = \cos\frac{\theta}{2}$, $\left|z + z^2\right| = \overline{OC} = 2\overline{OD} = 2\cos\frac{\theta}{2}$.

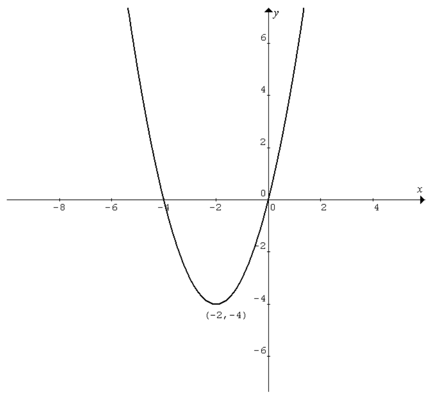
Q2div $\operatorname{Re} z + \operatorname{Re} z^2 = \cos\theta + \cos 2\theta$

$\operatorname{Re}\left(z + z^2\right) = \left|z + z^2\right|\cos\frac{3\theta}{2} = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$

Since $\operatorname{Re} z + \operatorname{Re} z^2 = \operatorname{Re}\left(z + z^2\right)$,

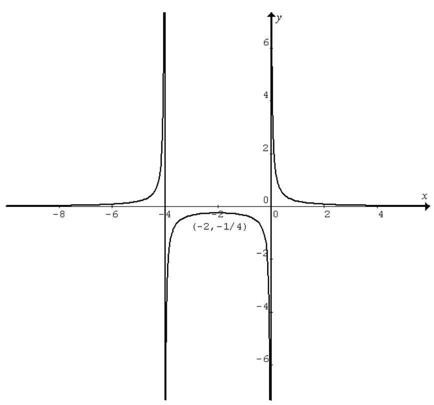
.: $\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$

Q3ai $y = x^2 + 4x = x(x+4)$

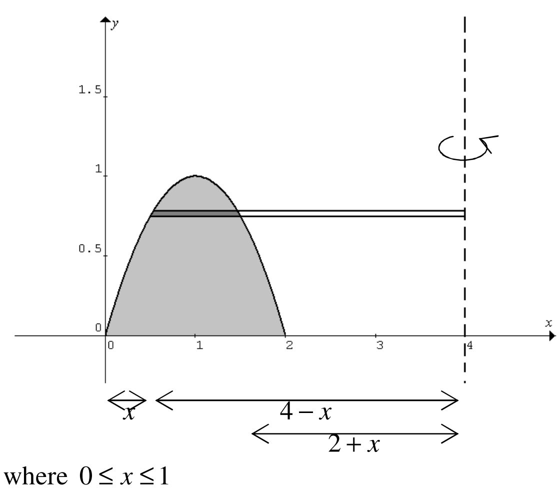


Q3aii $y = \frac{1}{x^2 + 4x} = \frac{1}{x(x+4)}$

Asymptotes: $x = 0$, $x = -4$, $y = 0$



Q3b



where $0 \le x \le 1$

$y = 2x - x^2$, $x = 1 - \sqrt{1-y}$

$V = \int_0^1 \left(\pi(4-x)^2 - \pi(2+x)^2\right)dy = 12\pi\int_0^1 (1-x)dy$

$= 12\pi\int_0^1 \sqrt{1-y}\,dy = 12\pi\left[-\frac{2(1-y)^{\frac{3}{2}}}{3}\right]_0^1 = 8\pi$

Q3c Let $\Pr(head) = p$ and $\Pr(tail) = 1-p$, given $p > 1-p$, i.e. $p > 0.5$

$\Pr(1head + 1tail) = \Pr(HT) + \Pr(TH) = 2p(1-p) = 0.48$

.: $p(1-p) = 0.24$, .: $p = 0.6$

$\Pr(2heads) = 0.6 \times 0.6 = 0.36$

Q3di Gradient $QA = \frac{c + \frac{c}{t}}{c + ct} = \frac{1}{t}$, .: gradient $\ell_1 = -t$

ℓ_1 is $y - \frac{c}{t} = -t(x - ct)$, i.e. $y = -tx + ct^2 + \frac{c}{t}$ ...... (1)

Q3dii Gradient $RA = \frac{c - \frac{c}{t}}{c - ct} = -\frac{1}{t}$, .: gradient $\ell_2 = t$

ℓ_2 is $y + \frac{c}{t} = t(x + ct)$, i.e. $y = tx + ct^2 - \frac{c}{t}$ ...... (2)

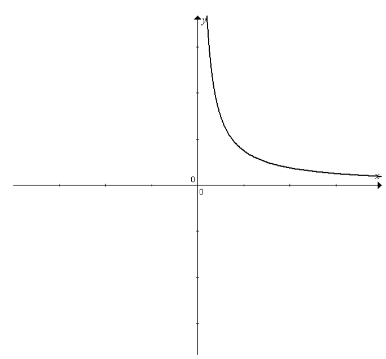
Q3diii Solve equations (1) and (2) simultaneously

$tx + ct^2 - \frac{c}{t} = -tx + ct^2 + \frac{c}{t}$, $2tx = 2\frac{c}{t}$, $x = \frac{c}{t^2}$, $y = ct^2$

Hence P is the point $\left(\frac{c}{t^2}, ct^2\right)$.

Q3div $x = \frac{c}{t^2}$, $y = ct^2$, .: $y = c\left(\frac{c}{x}\right)$, .: $xy = c^2$.

Since $x = \frac{c}{t^2} > 0$, the locus is $xy = c^2$ and $x \in R^+$, i.e. the half of the given rectangular hyperbola in the first quadrant.



Q4ai $\sqrt{x} + \sqrt{y} = 1$

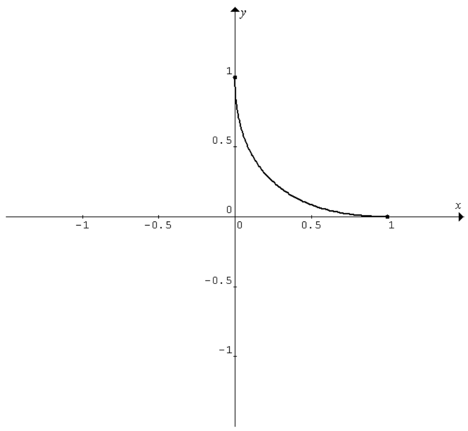
Implicit differentiation: $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$, .: $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

Q4aii $\sqrt{x} + \sqrt{y} = 1$ is defined for $x \in [0,1]$ and $y \in [0,1]$

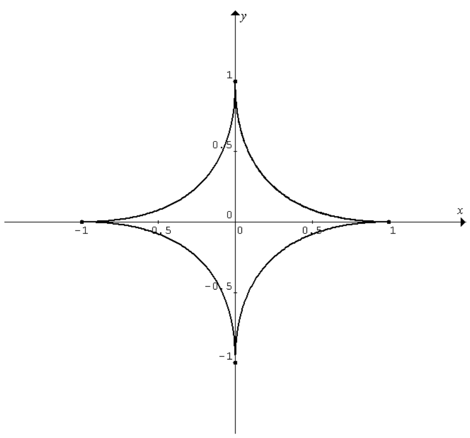
At $x = 0$, $y = 1$, $\frac{dy}{dx} \to -\infty$

At $y = 0$, $x = 1$, $\frac{dy}{dx} = 0$

$\frac{dy}{dx}$ is negative for $x \in (0,1)$



Q4aiii



Q4bi Resolving forces vertically:
$N\cos\alpha + F\sin\alpha - mg = 0$ ...... (1)
Resolving forces horizontally:
$N\sin\alpha - F\cos\alpha = \frac{mv^2}{r}$ ...... (2)
From (1), $N = \frac{mg - F\sin\alpha}{\cos\alpha}$ ...... (3)
Substitute (3) in (2), $\left(\frac{mg - F\sin\alpha}{\cos\alpha}\right)\sin\alpha - F\cos\alpha = \frac{mv^2}{r}$
Simplify to $F = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$

Q4bii $mg\sin\alpha - \frac{mv^2}{r}\cos\alpha = 0$, $v = \sqrt{gr\tan\alpha}$

Q4c Show $k \ge 4$ if $\frac{1}{a} + \frac{1}{b} = \frac{k}{a+b}$ for $a, b \in R^+$.
$\frac{1}{a} + \frac{1}{b} = \frac{k}{a+b}$, .: $(a+b)^2 = kab$ and $k > 0$
Let $a = nb$ where $n \in R^+$.
.: $(nb+b)^2 = knb^2$, $(n+1)^2 = kn$, $n^2 + (2-k)n + 1 = 0$
$n \in R^+$ exists when $\Delta = (2-k)^2 - 4 \ge 0$, .: $k \ge 4$ if $k > 0$.

Q4di Select 4 from 12: ${}^{12}C_4 = 495$

Q4dii There are ${}^{12}C_4$ ways to form the first group. For each of these there are ${}^{8}C_4$ ways to form the second group and one way to form the last group. The number of ways in this counting method is over counted by a factor of 3! because it takes into account the ordering of the groups.
Hence the number of ways $= \frac{{}^{12}C_4 \times {}^{8}C_4}{3!} = 5775$

Q5ai $B(b\cos\theta, b\sin\theta)$

Q5aii $P(a\cos\theta, b\sin\theta)$, $x = a\cos\theta$, $y = b\sin\theta$
.: $\frac{x}{a} = \cos\theta$ and $\frac{y}{b} = \sin\theta$
.: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$

Q5aiii At $P(a\cos\theta, b\sin\theta)$, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$
Gradient of the tangent at $P = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b\cos\theta}{a\sin\theta}$
Equation of the tangent at *P*:
$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$
$(a\sin\theta)y - ab\sin^2\theta = -(b\cos\theta)x + ab\cos^2\theta$
$(b\cos\theta)x + (a\sin\theta)y = ab$ ...... (1)

Q5aiv Change *b* to *a* in (1) to obtain the equation of the tangent to the circle at *A*:
$(a\cos\theta)x + (a\sin\theta)y = a^2$ ...... (2)
Eliminate *x* from (1) and (2): *a*(1) □ *b*(2)
$(a^2\sin\theta)y - (ab\sin\theta)y = 0$, .: $(a^2\sin\theta - ab\sin\theta)y = 0$
Since $a > b$, .: $a^2\sin\theta - ab\sin\theta \ne 0$ and $y = 0$
.: the two tangents intersect at a point on the *x*-axis.

Q5b $\int \frac{dy}{y(1-y)} = \int \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \ln y - \ln(1-y) + c$
$= \ln\left(\frac{y}{1-y}\right) + c$ where $0 < y < 1$

Q5ci $\frac{dy}{dx} = ay(1-y)$ is an inverted parabola and its horizontal axis intercepts are $(0,0)$ and $(1,0)$. Its axis of symmetry is $y = \frac{0+1}{2} = \frac{1}{2}$. .: $\frac{dy}{dx}$ has its maximum value when $y = \frac{1}{2}$.

Q5cii $\frac{dy}{dx} = ay(1-y)$, $\frac{dx}{dy} = \frac{1}{ay(1-y)}$, $x = \frac{1}{a}\int \frac{dy}{y(1-y)}$

.: $ax = \ln\left(\frac{y}{1-y}\right) + c$, $\frac{y}{1-y} = e^{ax-c}$, $y = (1-y)e^{ax-c}$,

$(1+e^{ax-c})y = e^{ax-c}$, .: $y = \frac{e^{ax-c}}{1+e^{ax-c}} = \frac{1}{e^{-ax+c}+1}$

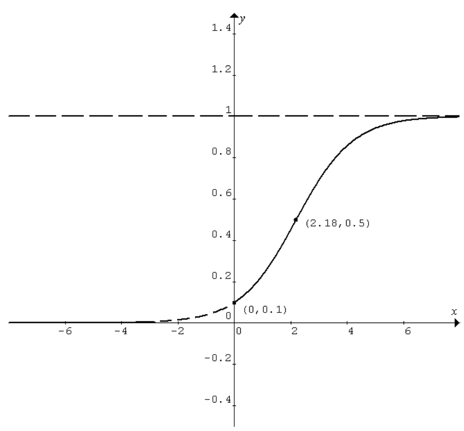
.: $y = \frac{1}{ke^{-ax}+1}$ where $k = e^c$

Q5ciii Given $y(0) = 0.1$, .: $y = \frac{1}{k+1} = 0.1$, .: $k = 9$

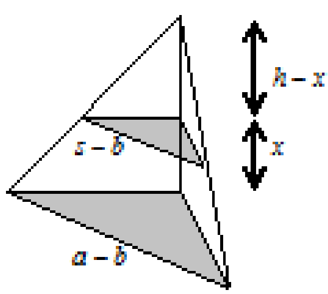
Q5civ As x increases from 0, the gradient of $y = \frac{1}{ke^{-ax}+1}$ increases to a maximum value at $y = \frac{1}{2}$ and then decreases,

.: there is a point of inflection at $y = \frac{1}{2}$.

Q5cv $y = \frac{1}{9e^{-ax}+1}$, $x \ge 0$



Q6ai Place two sloping edges together:



Two similar triangular solids: $\frac{s-b}{a-b} = \frac{h-x}{h} = 1 - \frac{x}{h}$,

.: $s - b = a - b - \frac{(a-b)}{h}x$, $s = a - \frac{(a-b)}{h}x$

Q6aii At height x m, area of horizontal layer

$= s^2 = \left(a - \frac{(a-b)}{h}x\right)^2$ m^2

Volume of frustum $= \int_0^h \left(a - \frac{(a-b)}{h}x\right)^2 dx$

$= \left[-\frac{h}{3(a-b)}\left(a - \frac{(a-b)}{h}x\right)^3\right]_0^h$

$= -\frac{h}{3(a-b)}(b^3 - a^3) = \frac{h}{3(a-b)}(a^3 - b^3)$

$= \frac{h}{3}(a^2 + ab + b^2)$ m^3

Q6b $a_n = 2a_{n-1} + a_{n-2}$, $n \ge 2$, $a_0 = a_1 = 2$

$a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$ is true when $n = 0$ because

$a_0 = (1+\sqrt{2})^0 + (1-\sqrt{2})^0 = 2$

It is also true when $n = 1$ because $a_1 = (1+\sqrt{2})^1 + (1-\sqrt{2})^1 = 2$

Assuming that it is true when $n = k-1$ and $n = k-2$, where $k \ge 2$, then $a_k = 2a_{k-1} + a_{k-2}$

$= 2(1+\sqrt{2})^{k-1} + 2(1-\sqrt{2})^{k-1} + (1+\sqrt{2})^{k-2} + (1-\sqrt{2})^{k-2}$

$= 2(1+\sqrt{2})^{k-1} + (1+\sqrt{2})^{k-2} + 2(1-\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-2}$

$= (1+\sqrt{2})^{k-2}\left(2(1+\sqrt{2})+1\right) + (1-\sqrt{2})^{k-2}\left(2(1-\sqrt{2})+1\right)$

$= \frac{(1+\sqrt{2})^k}{3+2\sqrt{2}}(3+2\sqrt{2}) + \frac{(1-\sqrt{2})^k}{3-2\sqrt{2}}(3-2\sqrt{2})$

$= (1+\sqrt{2})^k + (1-\sqrt{2})^k$

.: it is also true for $n = k$

.: it is true for all $n \ge 0$

Q6ci

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2$

$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$

$- i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$

$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Q6cii $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

.: $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Q6ciii $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$ Let $\theta = \frac{\pi}{10}$,

$$16\left(\sin\left(\frac{\pi}{10}\right)\right)^5 - 20\left(\sin\left(\frac{\pi}{10}\right)\right)^3 + 5\sin\left(\frac{\pi}{10}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$\therefore\ 16\left(\sin\left(\frac{\pi}{10}\right)\right)^5 - 20\left(\sin\left(\frac{\pi}{10}\right)\right)^3 + 5\sin\left(\frac{\pi}{10}\right) - 1 = 0$$

$\therefore\ x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$

Q6civ $16x^5 - 20x^3 + 5x - 1 = (x-1)\left(16x^4 + bx^3 + cx^2 + dx + 1\right)$

$\therefore\ -b + c = -20$, $-c + d = 0$ and $-d + 1 = 5$

$\therefore\ d = -4$, $c = -4$ and $b = 16$

$\therefore\ p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

Q6cv $p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1 = \left(4x^2 + ax - 1\right)^2$

Compare the coefficients of *x*: $-2a = -4$, $a = 2$

Q6cvi $16x^5 - 20x^3 + 5x - 1 = (x-1)\left(4x^2 + 2x - 1\right)^2 = 0$

Since $x - 1 = \sin\left(\frac{\pi}{10}\right) - 1 \neq 0$, $\therefore\ 4x^2 + 2x - 1 = 0$

$\therefore\ x = \sin\left(\frac{\pi}{10}\right) = \frac{-2 + \sqrt{4 + 16}}{8} = \frac{-1 + \sqrt{5}}{4}$ (first quadrant)

Q7ai $\angle ABD = \angle ACD$ (angles subtended on the circumference by the same arc are equal)

$\angle ADK = \angle CDB$ (given)

$\therefore\ \angle ADK + \angle KDB = \angle KDB + \angle CDB$

$\therefore\ \angle ADB = \angle KDC$

$\therefore\ \Delta ADB, \Delta KDC$ are similar

Q7aii $\Delta ADB, \Delta KDC$ are similar

$\therefore\ \frac{DC}{BD} = \frac{KC}{AB}$, $\therefore\ BD \times KC = AB \times DC$ ...... (1)

$\Delta ADK, \Delta BDC$ are similar

$\therefore\ \frac{AD}{BD} = \frac{AK}{BC}$, $\therefore\ BD \times AK = AD \times BC$ ...... (2)

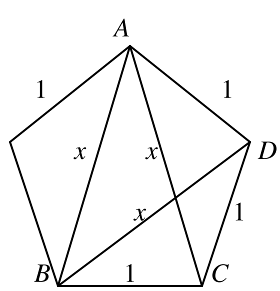
Add (1) and (2):

$BD \times AK + BD \times KC = AD \times BC + AB \times DC$

$BD \times (AK + KC) = AD \times BC + AB \times DC$

$\therefore\ BD \times AC = AD \times BC + AB \times DC$

Q7aiii

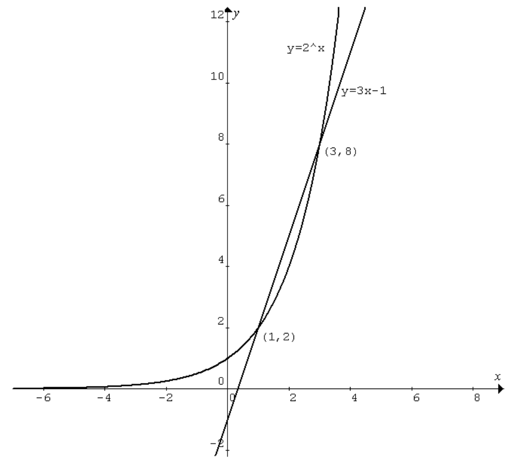


Since $BD \times AC = AD \times BC + AB \times DC$

$\therefore\ x^2 = 1 + x$ where $x > 0$, $x^2 - x - 1 = 0$

$$x = \frac{1 + \sqrt{1 + 4}}{2} = \frac{1 + \sqrt{5}}{2}$$

Q7b $2^x \geq 3x - 1$ for $x \geq 3$



Q7ci $P(x) = (n-1)x^n - nx^{n-1} + 1$, $n = 3,5,7,......$

Let $P'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = 0$

$\therefore\ P'(x) = n(n-1)x^{n-2}(x-1) = 0$

$\therefore\ x = 0$ or $x = 1$

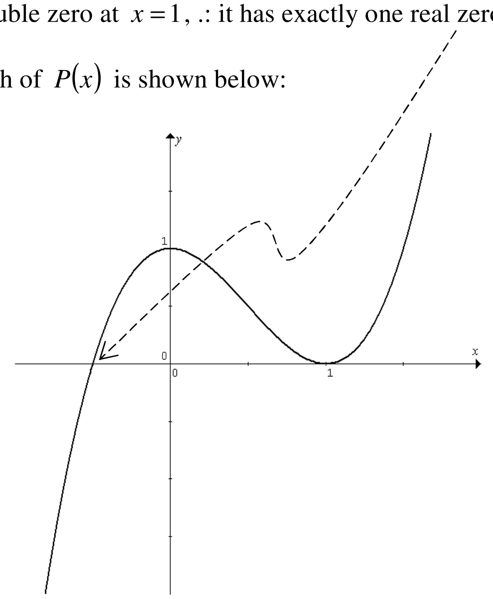
$\therefore\ P(x)$ has exactly two stationary points.

Q7cii $P(1) = n - 1 - n + 1 = 0$ and $(1,0)$ is a stationary point,

$\therefore\ P(x)$ has a double zero at $x = 1$

Q7ciii $P(x)$ is an odd-degree polynomial with 2 turning points and a double zero at $x = 1$, $\therefore$ it has exactly one real zero other than 1.

The graph of $P(x)$ is shown below:



Q7civ $P(x)=(n-1)x^n - nx^{n-1}+1$, $n=3,5,7,......$

For $n=3$:

$P(x)=2x^3-3x^2+1=(x-1)^2(2x+1)$, .: $\alpha=-\frac{1}{2}$

For $n>3$:

$P(-1)=(n-1)(-1)^n - n(-1)^{n-1}+1=-n+1-n+1=2-2n<0$

$P\left(-\frac{1}{2}\right)=(n-1)\left(-\frac{1}{2}\right)^n - n\left(-\frac{1}{2}\right)^{n-1}+1=-\frac{n-1}{2^n}-\frac{n}{2^{n-1}}+1$

$=\frac{-n+1-2n}{2^n}+1=-\frac{3n-1}{2^n}+1$

From part b, for $n>3$, $2^n>3n-1$, .: $1>\frac{3n-1}{2^n}$, .: $P\left(-\frac{1}{2}\right)>0$

.: $P(x)$ has an x-intercept in $\left(-1,-\frac{1}{2}\right]$.

Hence $-1<\alpha\le -\frac{1}{2}$.

Q7cv $4x^5-5x^4+1$ is in the form $P(x)=(n-1)x^n-nx^{n-1}+1$.

It has a real zero α between -1 and $-\frac{1}{2}$, and a double zero 1.

Each of these zeros has modulus ≤ 1. The polynomial is of degree 5 with real coefficients, .: it also has a pair of complex zeros which are conjugates, they are $a+ib$ and $a-ib$.

Product of the zeros: $\alpha\times 1^2\times(a+ib)(a-ib)=-\frac{1}{4}$

.: $a^2+b^2=-\frac{1}{4\alpha}$

Since $-1<\alpha\le-\frac{1}{2}$, $-4<4\alpha\le-2$, $2\le-4\alpha<4$,

$\frac{1}{4}<-\frac{1}{4\alpha}\le\frac{1}{2}$, .: $\frac{1}{4}<a^2+b^2\le\frac{1}{2}$, $\frac{1}{2}<\sqrt{a^2+b^2}\le\frac{1}{\sqrt{2}}$,

.: $\frac{1}{2}<|a\pm ib|\le\frac{1}{\sqrt{2}}$

Hence each of the zeros of $4x^5-5x^4+1$ has modulus ≤ 1.

Q8a Apply the reduction formula

$\int\cos^n x\,dx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x\,dx$ to A_n

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\left[\frac{1}{2n}\cos^{2n-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2n-1}{2n}\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,dx$

$=\frac{2n-1}{2n}A_{n-1}$

.: $nA_n=\frac{2n-1}{2}A_{n-1}$ for $n\ge 1$

OR integration by parts on A_n: $\left(\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx\right)$

Let $u=\cos^{2n-1}x$, $\frac{du}{dx}=-(2n-1)\sin x\cos^{2n-2}x$

Let $\frac{dv}{dx}=\cos x$, $v=\sin x$

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$

$=\left[\sin x\cos^{2n-1}x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x\left(-(2n-1)\sin x\cos^{2n-2}x\right)dx$

$=(2n-1)\int_0^{\frac{\pi}{2}}(1-\cos^2 x)\cos^{2n-2}x\,dx$

$=(2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,dx-(2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$

.: $A_n=(2n-1)A_{n-1}-(2n-1)A_n$, .: $nA_n=\frac{2n-1}{2}A_{n-1}$ for $n\ge1$.

Q8b Integration by parts on $A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$:

Let $u=\cos^{2n}x$, $\frac{du}{dx}=-2n\sin x\cos^{2n-1}x$

Let $\frac{dv}{dx}=1$, $v=x$

$A_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\left[x\cos^{2n}x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}x\left(-2n\sin x\cos^{2n-1}x\right)dx$

$=2n\int_0^{\frac{\pi}{2}}x\sin x\cos^{2n-1}x\,dx$

Q8c Let $\frac{dv}{dx}=x$, $v=\frac{x^2}{2}$

Let $u=\sin x\cos^{2n-1}x$,

$\frac{du}{dx}=\cos x\cos^{2n-1}x+\sin x(2n-1)\cos^{2n-2}x(-\sin x)$

$=\cos^{2n}x-(2n-1)(1-\cos^2 x)\cos^{2n-2}x$

$=\cos^{2n}x-(2n-1)\cos^{2n-2}x+(2n-1)\cos^{2n}x$

$=2n\cos^{2n}x-(2n-1)\cos^{2n-2}x$

.: $A_n=2n\int_0^{\frac{\pi}{2}}x\sin x\cos^{2n-1}x\,dx$

$=2n\left[\frac{x^2}{2}\sin x\cos^{2n-1}x\right]_0^{\frac{\pi}{2}}-2n\int_0^{\frac{\pi}{2}}\frac{x^2}{2}\left(2n\cos^{2n}x-(2n-1)\cos^{2n-2}x\right)dx$

$=n(2n-1)\int_0^{\frac{\pi}{2}}x^2\cos^{2n-2}x\,dx-2n^2\int_0^{\frac{\pi}{2}}x^2\cos^{2n}x\,dx$

.: $A_n=n(2n-1)B_{n-1}-2n^2B_n$.: $\frac{A_n}{n^2}=\frac{(2n-1)}{n}B_{n-1}-2B_n$

Q8d From part c:

$$\frac{A_n}{n^2}=\frac{(2n-1)}{n}B_{n-1}-2B_n\ ,\ \therefore\ \frac{1}{n^2}=\frac{(2n-1)}{n}\frac{B_{n-1}}{A_n}-2\frac{B_n}{A_n}\ ......\ (1)$$

From part a: $nA_n=\frac{(2n-1)}{2}A_{n-1}$, $\therefore\ \frac{2}{A_{n-1}}=\frac{(2n-1)}{nA_n}$ ...... (2)

Substitute (2) in (1): $\frac{1}{n^2}=2\left(\frac{B_{n-1}}{A_{n-1}}-\frac{B_n}{A_n}\right)$ for $n\ge 1$

Q8e

$$\sum_{k=1}^{n}\frac{1}{k^2}=2\left(\frac{B_0}{A_0}-\frac{B_1}{A_1}\right)+2\left(\frac{B_1}{A_1}-\frac{B_2}{A_2}\right)+2\left(\frac{B_2}{A_2}-\frac{B_3}{A_3}\right)+...+2\left(\frac{B_{n-1}}{A_{n-1}}-\frac{B_n}{A_n}\right)$$

$$=2\left(\frac{B_0}{A_0}-\frac{B_n}{A_n}\right)$$

where $A_0=\int_0^{\frac{\pi}{2}}dx=[x]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$ and $B_0=\int_0^{\frac{\pi}{2}}x^2dx=\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}=\frac{\pi^3}{24}$

$$\therefore\ \sum_{k=1}^{n}\frac{1}{k^2}=2\left(\frac{B_0}{A_0}-\frac{B_n}{A_n}\right)=2\frac{B_0}{A_0}-2\frac{B_n}{A_n}=\frac{\pi^2}{6}-2\frac{B_n}{A_n}$$

Q8f $\sin x\ge\frac{2x}{\pi}$, $\sin^2 x=1-\cos^2 x\ge\frac{4x^2}{\pi^2}$

$\therefore\ \cos^2 x\le 1-\frac{4x^2}{\pi^2}$, $\therefore\ \cos^{2n}x\le\left(1-\frac{4x^2}{\pi^2}\right)^n$ for $0\le x\le\frac{\pi}{2}$

Hence $B_n=\int_0^{\frac{\pi}{2}}x^2\cos^{2n}x\,dx\le\int_0^{\frac{\pi}{2}}x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx$

Q8g Integration by parts:

Let $u=x$ and $\frac{dv}{dx}=x\left(1-\frac{4x^2}{\pi^2}\right)^n$

$\therefore\ \frac{du}{dx}=1$ and

$$v=\int x\left(1-\frac{4x^2}{\pi^2}\right)^n dx=\int\frac{-\pi^2p^n}{8}dp=-\frac{\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}$$

where $p=1-\frac{4x^2}{\pi^2}$.

$$\int_0^{\frac{\pi}{2}}x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx$$

$$=\left[-\frac{\pi^2}{8(n+1)}x\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\frac{\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx$$

$$=\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx$$

Q8h Let $x=\frac{\pi}{2}\sin t$.

$$\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx=\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)^{n+1}\frac{\pi}{2}\cos t\,dt$$

$$=\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\left(\cos^2 t\right)^{n+1}\cos t\,dt=\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt$$

$\therefore\ B_n\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt$ from parts (f) and (g).

For $0\le t\le\frac{\pi}{2}$, $0\le\cos^3 t\le 1$

$$\therefore\ B_n\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n}t\,dt$$

$$\therefore\ B_n\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\,dt\le\frac{\pi^3}{16(n+1)}A_n$$

Q8i Note that $A_n>0$ and $B_n>0$

From part (h), $2\frac{B_n}{A_n}\le\frac{\pi^3}{8(n+1)}$

From part (e), $\frac{\pi^2}{6}-2\frac{B_n}{A_n}=\sum_{k=1}^{n}\frac{1}{k^2}$

$$\therefore\ \frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}\le\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}$$

Q8j The value of $\sum_{k=1}^{n}\frac{1}{k^2}$ lies between $\frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}$ and $\frac{\pi^2}{6}$.

As $n\to\infty$, $\frac{\pi^3}{8(n+1)}\to 0$, $\sum_{k=1}^{n}\frac{1}{k^2}\to\frac{\pi^2}{6}$

$$\therefore\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}$$

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