itute.com

Online & home tutors Registered business name: itute ABN: 96 297 924 083

Specialist Mathematics

2010

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 When the quadratic equation $z^2 - z + 1 = 0$ is solved over C,

A. it has no solutions.

B. the solutions are $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

C. the solutions are $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

D. the solutions are $z = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i$.

E. the solutions are $z = \frac{1}{2} \pm \frac{\sqrt{5}}{2} i$.

Question 2 Given $z = -cis(1)$,

A. $|z| = -1$ and $Arg(z) = 1$.

B. $|z| = 1$ and $Arg(z) = -1$.

C. $|z| = 1$ and $Arg(z) = 1 - \pi$.

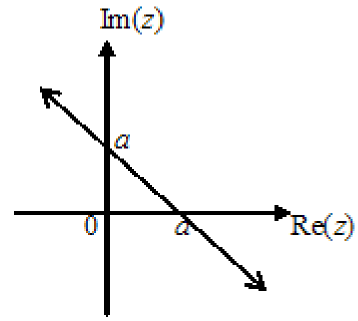
D. $|z| = -1$ and $Arg(z) = \pi$.

E. $|z| = 1$ and $Arg(z) = \pi - 1$.

Question 3 The polynomial $P(z) = z^3 + 2iz^2 + 2z + 4i$ has

A. no real solutions.
B. a pair of conjugate roots.
C. three linear factors over C.
D. three solutions over C.
E. two real solutions and a complex solution.

Question 4 The straight line on the complex plane shown on the left can be defined by



A. $\left\{z : Arg(z) = \frac{3\pi}{4}\right\}$.
B. $\{z : |z - a - ai| - |z| = 0\}$.
C. $\{z : |z - a + ai| = |z|\}$.
D. $\{z : |z + a - ai| - |z| = 0\}$.
E. $\{z : |z + a + ai| = |z|\}$.

Question 5 The graph of $y = x + \frac{b}{x}$, where $b \in R \setminus \{0\}$,

A. always has two stationary points.
B. always has two asymptotes.
C. has R as its domain.
D. always has R as its range.
E. has a y-intercept.

Question 6 The graph of $y = \frac{2}{4x^2 + px + q^2}$, where $q > 0$, has a stationary point when

A. $p < -4q$.
B. $p < 0$.
C. $p \geq 0$.
D. $p > q$.
E. $p < q$.

Question 7 $\sin\left(a + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) =$

A. $\frac{1}{2}\cos a - \frac{\sqrt{3}}{2}\sin a$.
B. $-\frac{1}{2}\cos a + \frac{\sqrt{3}}{2}\sin a$.
C. $\frac{1}{2}\sin a - \frac{\sqrt{3}}{2}\cos a$.
D. $-\frac{1}{2}\sin a + \frac{\sqrt{3}}{2}\cos a$.
E. $\sin a + \frac{1}{2}$.

Question 8 The domain of the function $f(x)=\sin^{-1}\left(\frac{x}{a}+b\right)+c$, where $a,b,c\in(-\infty,0)$, is

A. $[-a(1+b),\ a(1-b)]$.
B. $[-a-b,\ a-b]$.
C. $[-a(1+b)-c,\ a(1-b)-c]$.
D. $[a-b,\ -a-b]$.
E. $[a(1-b),\ -a(1+b)]$.

Question 9 Let $\tan^{-1}a=0.3$ and $\tan^{-1}b=0.2$. In terms of *a* and/or *b*, $\tan(0.1)=$

A. $\frac{a-b}{ab-1}$.
B. $\frac{a-b}{1-ab}$.
C. $\frac{\sqrt{1+b^2}-1}{b}$.
D. $\frac{\sqrt{1+b^2}+1}{b}$.
E. $a-b$.

Question 10 When $\pi<\theta<\frac{3\pi}{2}$, $\cos^{-1}(\cos\theta)=$

A. θ.
B. $\pi-\theta$.
C. $2\pi-\theta$.
D. $\theta-2\pi$.
E. $\theta-\pi$.

Question 11 $-2\tilde{i}+3\tilde{k}$, $\tilde{j}-2\tilde{k}$ and which one of the following vectors are linearly independent?

A. $2\tilde{i}-\tilde{j}-\tilde{k}$
B. $4\tilde{i}-3\tilde{j}$
C. $3\tilde{i}-4\tilde{j}$
D. $6\tilde{i}-2\tilde{j}-5\tilde{k}$
E. $2\tilde{i}-2\tilde{j}+\tilde{k}$

Question 12 *A*, *B* and *P* are points on a **unit** circle centred at *O*. Given $\overrightarrow{AO}.\overrightarrow{BO}=\frac{1}{2}$ and $\overrightarrow{AP}.\overrightarrow{BP}=\sqrt{3}$, the value of $|\overrightarrow{AP}||\overrightarrow{BP}|$ is

A. 0.6 B. 0.8 C. 1 D. 2 E. 3

Question 13 Point Q divides the line segment joining point $P(-1,0,2)$ and point $R(2,1,-2)$ into a ratio of 2 : 3. The distance of point Q from the origin $O(0,0,0)$ is

A. $\dfrac{\sqrt{29}}{5}$.
B. $\sqrt{29}$.
C. $\dfrac{\sqrt{29}}{6}$.
D. $\dfrac{1}{2}$.
E. $\dfrac{3}{5}$.

Question 14 The angles that $\dfrac{\sqrt{3}}{2}\underset{\sim}{i} - \dfrac{1}{\sqrt{2}}\underset{\sim}{j} - \dfrac{1}{2}\underset{\sim}{k}$ makes with the x, y and z axes are respectively

A. 30°, 45° and 60°.
B. 30°, 135° and 120°.
C. 30°, –45° and –60°.
D. $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$, $-\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ and $-\cos^{-1}\left(\dfrac{1}{\sqrt{6}}\right)$.
E. $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$, $\cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$ and $\cos^{-1}\left(-\dfrac{1}{\sqrt{6}}\right)$.

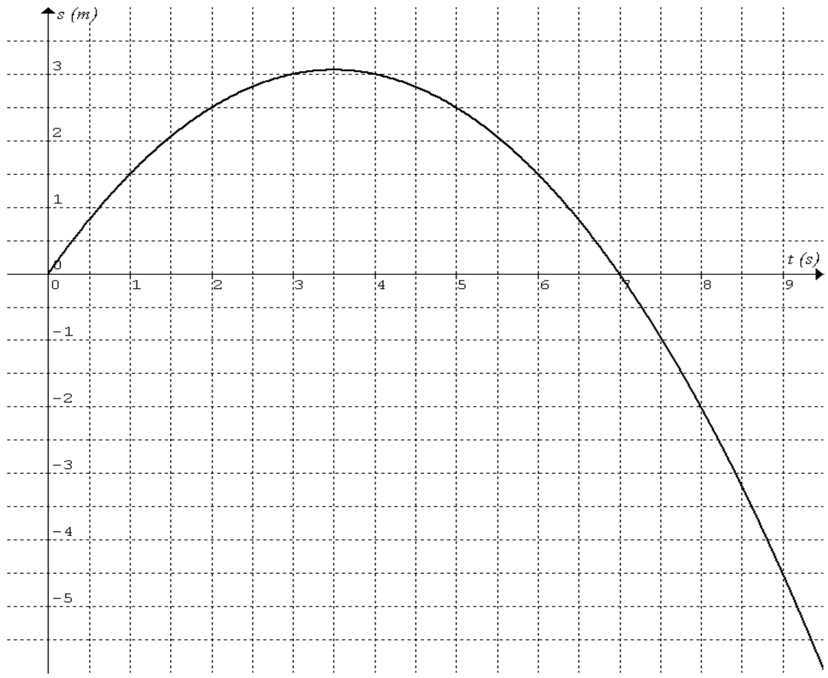
Question 15 Which one of the following statements is **true** about a particle with velocity $\underset{\sim}{v} = 2\cos^{-1}(t)\underset{\sim}{i} - 3\cos^{-1}(t)\underset{\sim}{j} + \cos^{-1}(t)\underset{\sim}{k}$, $0 \le t \le 1$.

A. The speed of the particle increases with time t.
B. The particle moves in a straight line.
C. The particle moves with constant acceleration.
D. The distance of the particle from its initial position decreases with time t.
E. The initial speed of the particle is $\sqrt{14}$.

Question 16 If $\dfrac{dy}{dh} = \sqrt{y(2-y)}\dfrac{dx}{dh}$, y may be expressed in terms of x as

A. $y = \sin(x+2)+1$
B. $y = \sin(x+3)-1$
C. $y = \sin(x-1)$
D. $y = \sin(2x-1)+1$
E. $y = \sin(3x-1)-1$

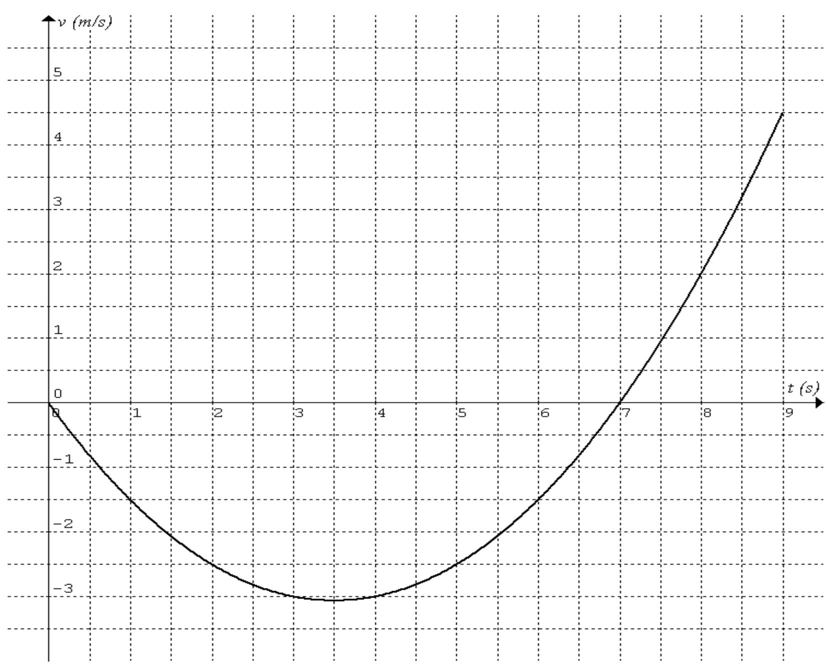
Question 17 The ***displacement***-time graph of a particle travelling along the x-axis is shown below.



Initially the particle is at $x = 2$. The particle is at $x = 0$ when

A. $t = 0$. B. $t = 1.4$. C. $t = 5.6$. D. $t = 7$. E. $t = 8$.

Question 18 The velocity-time graph ($0 \le t \le 9$) of a particle is shown below.



The particle is furthest from its initial position when

A. $t = 0$. B. $t = 3.5$. C. $t = 7$. D. $t = 9$. E. its acceleration is zero.

Question 19 Using a suitable substitution, $\int_0^{\frac{\pi}{2}} \cos^3 x\sqrt{1-\sin x}\,dx$ may be expressed completely in terms of u as

A. $\int_0^1 \left(2u^{\frac{3}{2}} - u^{\frac{5}{2}}\right)du$

B. $\int_0^1 \left(-2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$

C. $\int_0^{\sin^{-1}(1)} \left(2u^{\frac{3}{2}} - u^{\frac{5}{2}}\right)du$

D. $\int_0^{\sin^{-1}(1)} \left(-2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$

E. $\int_{\cos^{-1}(0)}^1 \left(-2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$

Question 20 An object slides down an inclined plane ($\mu = 0.2$). The object moves at constant speed when the inclined plane is at an angle of $\theta°$ with the horizontal. The value of θ

A. depends on the mass of the object.
B. depends on the speed of the object.
C. on earth is different from the value of θ on the moon.
D. is approximately 11.
E. is indeterminable without further information.

Question 21 A particle moves along the x-axis. Its velocity is given by $v = -\sqrt{100 - x}$. Which one of the following statements is **NOT** true?

A. The particle has a constant acceleration.
B. The particle has a negative acceleration.
C. The particle speeds up.
D. The particle slows down.
E. The particle continues to move in the same direction.

Question 22 The floor of a lift is an inclined plane (10° to the horizontal). A 10-kg object is at rest on the floor whilst the lift is stationary. When the lift moves downwards with speed increasing at 1 m/s in a second, the reaction force of the floor on the object is

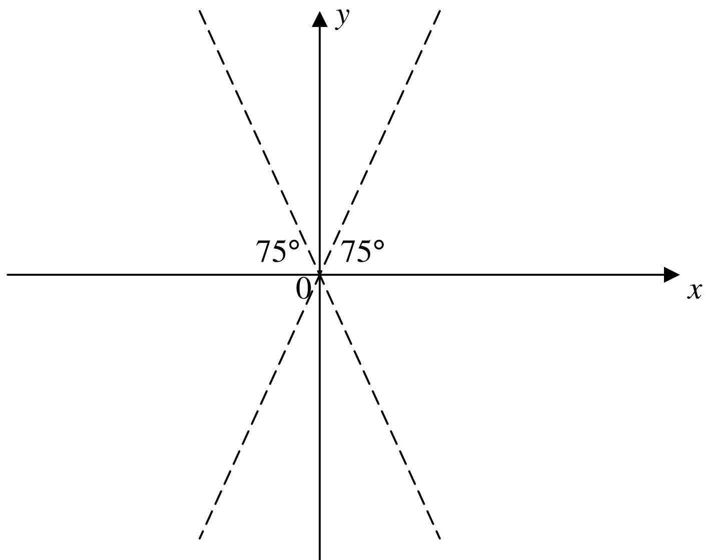
A. 86.7 N perpendicular to the floor.
B. 88 N vertically upward.
C. 88 N perpendicular to the floor.
D. 96.5 N perpendicular to the floor.
E. 98 N vertically upward.

SECTION 2 Extended-answer questions

Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1



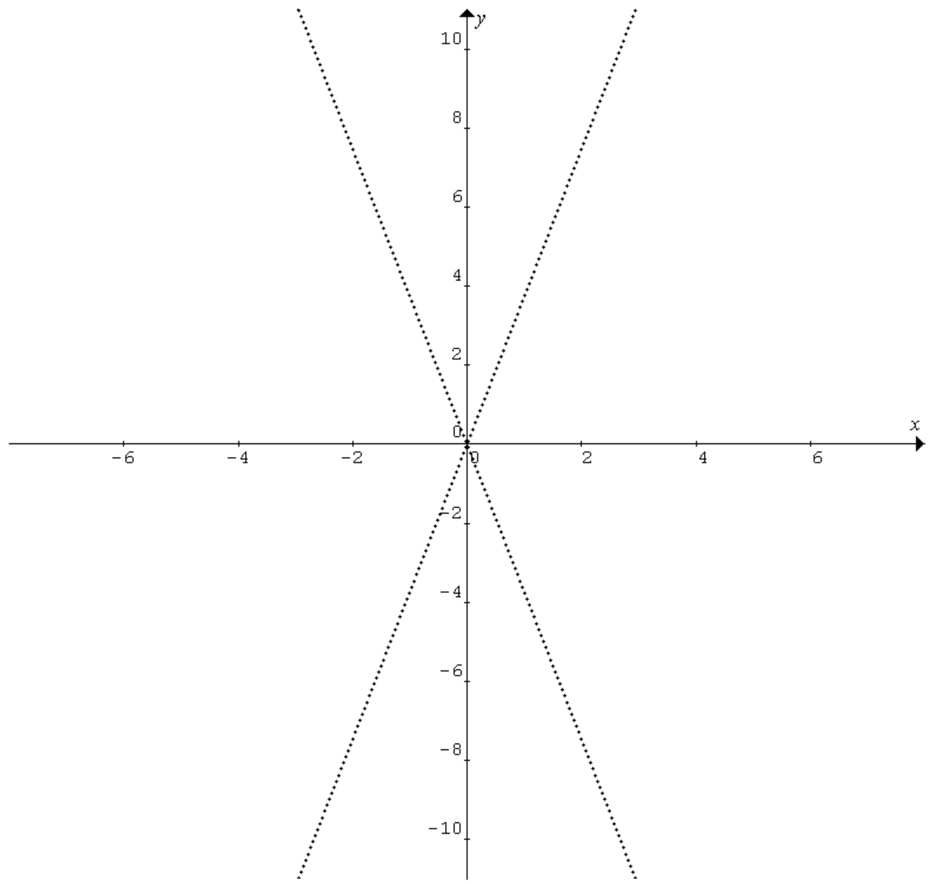
a. Show that the equations of the two dotted intersecting lines are $y = \pm\left(2+\sqrt{3}\right)x$. 2 marks

b. A hyperbola has the dotted lines as its asymptotes and $y = \pm\left(\sqrt{3}+1\right)$ as its *y*-intercepts. Determine the equation of the hyperbola.

2 marks

c. Sketch accurately the graph of the hyperbola showing the important features on the axes below.

1 mark



Let O be the origin $(0,0)$. $\overrightarrow{OA}$ is the position vector of moving point A which lies on the dotted line with positive gradient, and $\overrightarrow{OB}$ is the position vector of moving point B which lies on the dotted line with negative gradient. The unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are in the positive x and y directions respectively.

d i. Given the x-coordinates of A and B are a and b respectively, express $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of a, b, $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

2 marks

d ii. Hence determine the position vector $\overrightarrow{OM}$ in terms of a, b, $\underset{\sim}{i}$ and $\underset{\sim}{j}$, where M is the mid-point of $\overline{AB}$.

1 mark

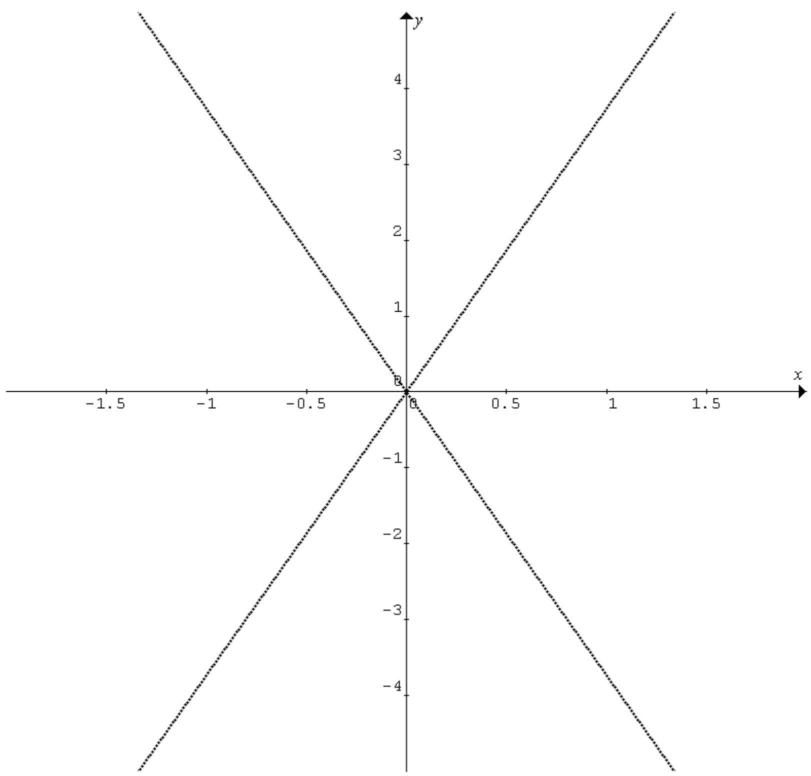
d iii. Given $\left|\overrightarrow{BA}\right| = 2$, show that $\left(2+\sqrt{3}\right)^2 (a+b)^2 + (a-b)^2 = 4$. 1 mark

d iv. Hence show that the locus of point M is an ellipse when a and b are varied and the condition $\left|\overrightarrow{BA}\right| = 2$ is satisfied.

2 marks

d v. Sketch accurately the graph of the ellipse showing the important features on the axes below.

1 mark



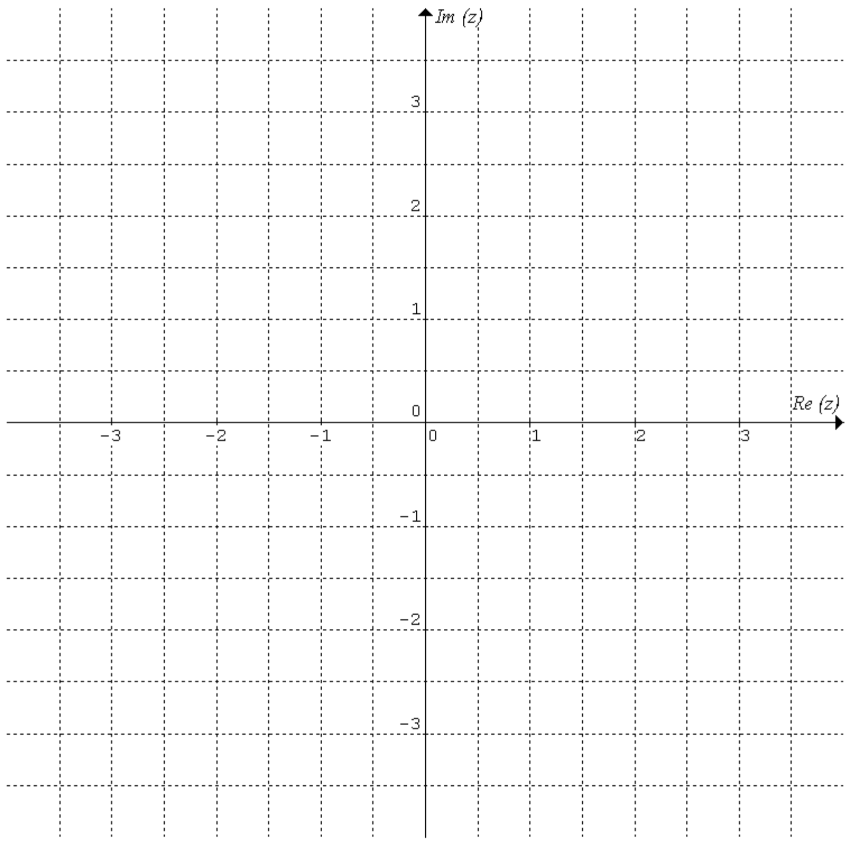
d vi. Find the values of b (correct to three decimal places) when $a = 0.500$. Hence draw accurately the line segment $\overline{AB}$ on the axes above, with B *in the second quadrant*. Mark the point M and label the ends of line segment $\overline{AB}$.

3 marks

Question 2 Consider $f(z)=z^3-6iz^2-12z+7i$.

a i. Express $f(z)$ in the form $(z-a)^3+b$, where $a,b\in C$. 2 marks

a ii. Sketch $\{z:|z-2i|=1\}$ on the argand diagram below. 1 mark



a iii. Hence sketch the roots of $f(z)=0$ on the argand diagram above. 2 marks

a iv. Determine the roots of $f(z)=0$ in exact $x+yi$ form. 2 marks

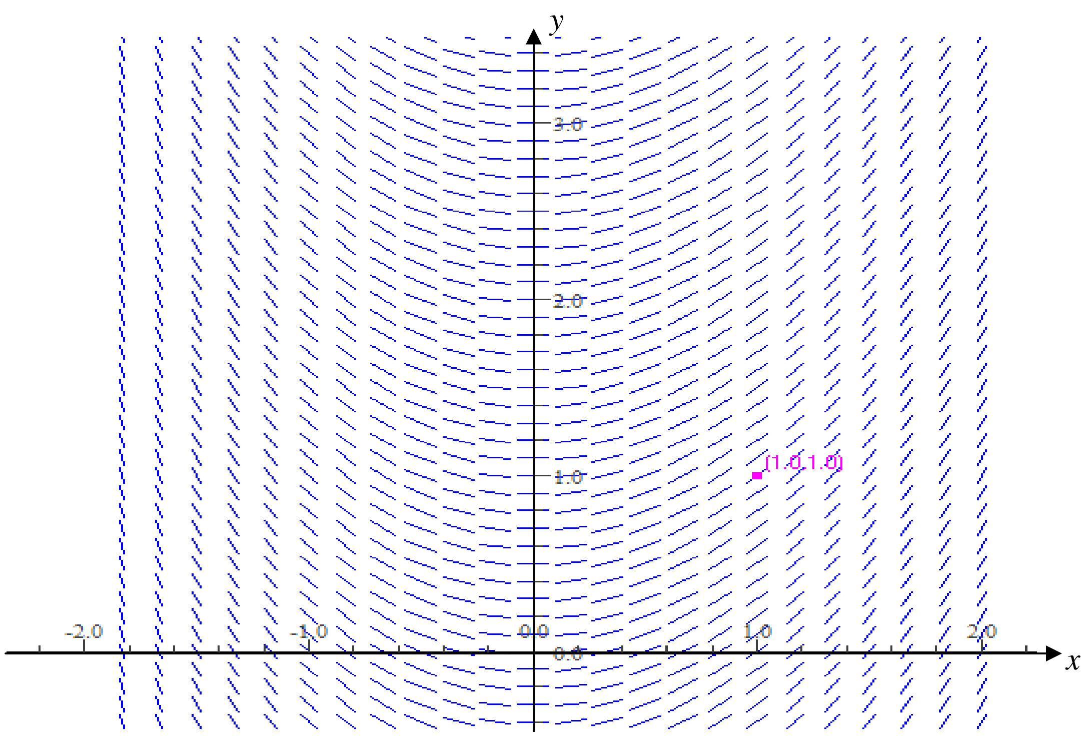
b. Determine the roots of $-iz^3 - 6z^2 + 12iz + 7 = 0$. 1 mark

c. Let $g(z) = f(z) - 7i$. Sketch the roots of $g(z) = 0$ on the argand diagram above. Describe the locations of the roots of $g(z) = 0$ in relation to the locations of the roots of $f(z) = 0$.

2 marks

Question 3

A slope field of the differential equation $\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right)$ is shown below. The graph of a particular solution of the differential equation passes through the point $(1,1)$.



a. State the maximal domain of the differential equation $\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right)$. 1 mark

b i. On the slope field above sketch the graph of the particular solution of the differential equation.

2 marks

b ii. Hence estimate the value of y (correct to one decimal place) when $x = 1.5$.

1 mark

c. Divide the section from $x = 1$ to $x = 1.5$ into two equal intervals. Given $y = 1$ when $x = 1$, by Euler's method (first-order approximation) find the approximate value of y (correct to two decimal places) when $x = 1.5$.

2 marks

d i. Given $y = 1$ when $x = 1$, a numerical solution of $\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right)$ when $x = 1.5$ can be found by evaluation of a definite integral. Write down this definite integral.

1 mark

d ii. Hence find the value of y (correct to two decimal places) when $x = 1.5$.

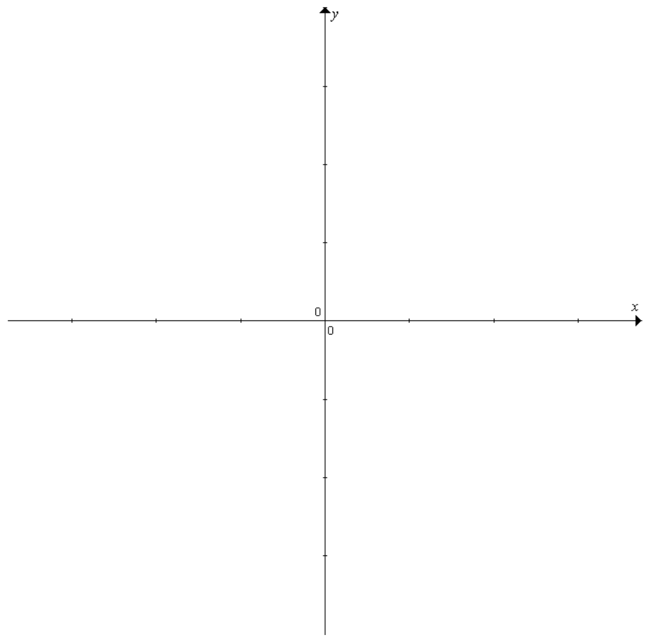
1 mark

Question 4

Consider $f(x)=\dfrac{x}{\sqrt{p^2-x^2}}+q$, where $p,q\in R^+$.

a. Sketch the graph of $f(x)$ on the axes below. Show and label the axis-intercept(s) and asymptote(s) in terms of *p* and *q*.

2 marks



Now let $p=\sqrt{3}$ and $q=\sqrt{3}$.

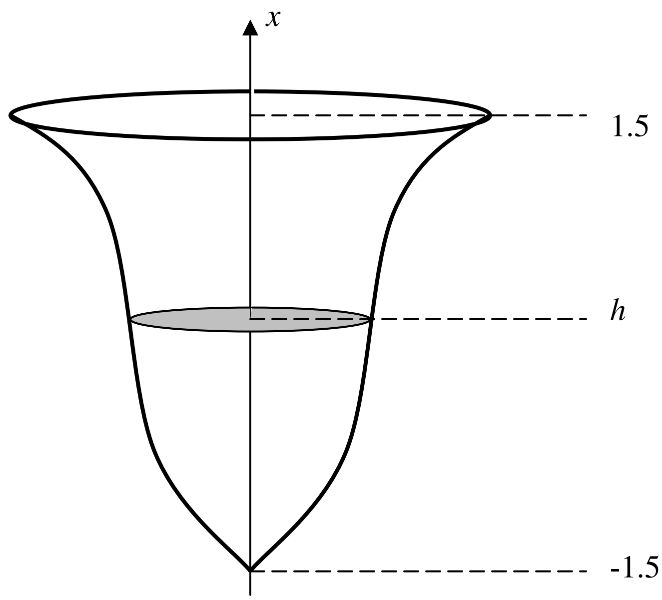
b. Without using CAS or calculator show that the area of the region bounded by the graph of $f(x)$, the *x*-axis and $x=\dfrac{3}{2}$ is $3\sqrt{3}$. 2 marks

c i. The region specified in part **b.** is rotated about the *x*-axis to form a solid of revolution with volume *V*. Write a definite integral for finding *V*.

1 mark

c ii. Find the value of *V* correct to one decimal place. 1 mark

A container (wall of negligible thickness) in the shape of the solid in **c i.** is filled with water at a rate of 0.5 cm^3 per second. Linear measurements are in cm.



d. Use calculus to find the volume of water (in terms of h in cm^3) in the container when $x = h$, where $-1.5 \le h \le 1.5$.

4 marks

e. Find the exact rate of increase (cm per second) in the depth of water when $h = 0$.

2 marks

Question 5

The speed of a cyclist (total mass 85 kg) travelling along a straight road is $v = \frac{1}{2}\tan^{-1}(5t + 2.5) + \frac{\pi}{2}$ m/s, $t \geq 0$.
A car (total mass 1200 kg) travels in the same direction at constant speed of 20 m/s. The car passes the cyclist at $t = 0$. Two seconds later the car slows down and its speed is given by $v = \frac{20}{1 + 0.5(t-2)^4}$ m/s, $t \geq 2$.

a. Determine the initial speed and acceleration of the cyclist correct to three decimal places. 3 marks

b. Determine the resultant force on the cyclist at $t = 0$. Give your answer correct to the nearest newton.

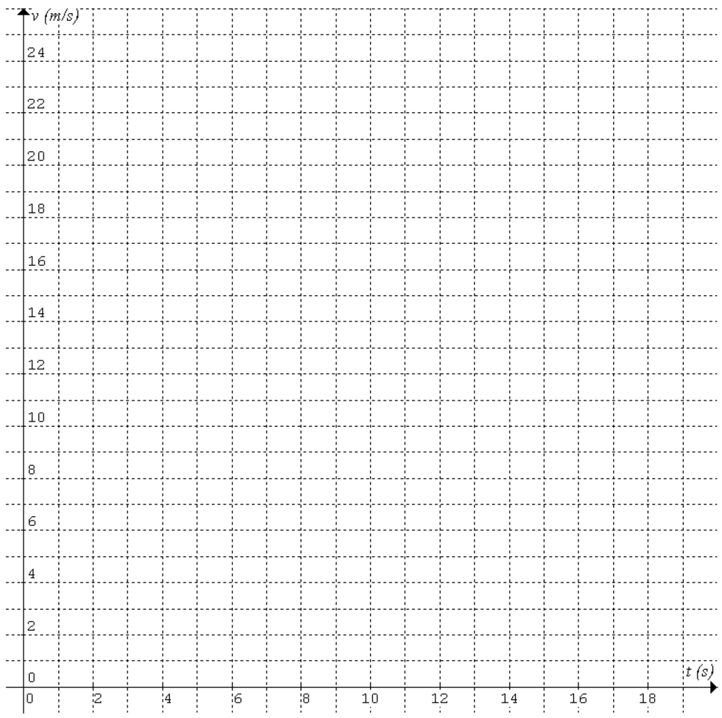
1 mark

c. Determine the difference in initial momentum of the cyclist and the car. Give your answer correct to the nearest kg m/s.

1 mark

d. Sketch the speed-time graphs for the cyclist and the car on the same axes below. Show and label intercept(s) and asymptote(s).

4 marks



e. Determine whether the cyclist or the car is ahead at $t = 10$ s. Find the distance between the cyclist and the car at $t = 10$ s. Give your answer correct to the nearest metre.

2 marks

f. Determine the time when the cyclist and the car are next to each other. Give your answer correct to the nearest second.

2 marks

End of Exam 2

