2011 NSW BOS Mathematics Extension 1 Solutions

Q1a $P\left(\frac{-1\times1+9\times4}{4+1},\frac{-2\times1+3\times4}{4+1}\right)=P(7,2)$

Q1b

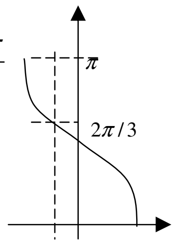
$$\frac{d}{dx}\left(\frac{\sin^2 x}{x}\right)=\frac{x(2\sin x\cos x)-\sin^2 x}{x^2}=\frac{\sin x(2x\cos x-\sin x)}{x^2}$$

Q1c $\frac{4-x}{x}<1$, $\frac{4}{x}-1<1$, $\frac{4}{x}<2$, $\frac{2}{x}<1$, .: $x<0$ or $x>2$

Q1d $u=\sqrt{x}$, $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$,

$$\int_1^4\frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\int_1^4 2e^u\frac{du}{dx}dx=\int_1^2 2e^u du=\left[2e^u\right]_1^2=2e(e-1)$$

Q1e $0<\cos^{-1}\left(-\frac{1}{2}\right)<\pi$, $\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$



Q1f $x^2\ge0$, $x^2+e\ge e$, $\ln\left(x^2+e\right)\ge\ln e$ since $\ln x$ is an increasing function. .: $\ln\left(x^2+e\right)\ge1$, range of $f(x)$ is $[1,\infty)$.

Q2a $P(x)=x^3-ax^2+x$

When $P(x)$ is divided by $x-3$, $R=P(3)=27-9a+3=12$,

.: $a=2$

When $P(x)$ is divided by $x+1$, $R=P(-1)=-1-2-1=-4$

Q2b $x_0=\frac{1}{2}$, $f(x)=\cos2x-x$, $f'(x)=-2\sin2x-1$

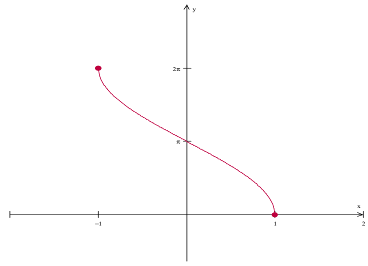
$f\left(\frac{1}{2}\right)=\cos1-\frac{1}{2}\approx0.0403$, $f'\left(\frac{1}{2}\right)=-2\sin1-1\approx-2.6829$

$x_1\approx x_0-\frac{f(x_0)}{f'(x_0)}=\frac{1}{2}-\frac{0.0403}{-2.6829}\approx0.52$

Q2c ${}^8C_r(3x)^{8-r}\left(-\frac{4}{x}\right)^r=ax^{8-2r}=ax^2$, .: $8-2r=2$, $r=3$

.: $a={}^8C_3\times3^5\times(-4)^3=-870912$ is the coefficient of x^2.

Q2d Domain: $[-1,1]$; range: $[0,2\pi]$



Q2ei *Number of arrangements of 40 different songs* $=40!$

Q2eii *Number of arrangements of 3 particular songs* $=3!$
Number of arrangements of 37 remaining songs $=37!$
Total number of arrangements $=3!37!$

Q3ai $x=A\cos nt+B\sin nt$, $\frac{dx}{dt}=-nA\sin nt+nB\cos nt$

$$\frac{d^2x}{dt^2}=-n^2A\cos nt-n^2B\sin nt=-n^2(A\cos nt+B\sin nt)=-n^2x$$

Q3aii At $t=0$, $x=A\cos0+B\sin0=0$,

$v=\frac{dx}{dt}=-nA\sin0+nB\cos0=2n$.

.: $A=0$ and $-A\sin0+B\cos0=2$, i.e. $B=2$

Q3aiii $x=2\sin nt=2$ when $nt=\frac{\pi}{2}$, $t=\frac{\pi}{2n}$ is the first time.

Q3aiv $t=\frac{2\pi}{n}$ is a period of the motion.

Total distance $=4\times2=8$ units

Q3bi $y=x^2$, $\frac{dy}{dx}=2x$. At $x=t$, $\frac{dy}{dx}=2t$.

Tangent at $P(t,t^2)$: $y-t^2=2t(x-t)$, $y=2tx-t^2$

Q3bii Tangent at $Q\left(1-t,(1-t)^2\right)$: $y=2(1-t)x-(1-t)^2$

Q3biii Given that $P(t,t^2)$ and $Q\left(1-t,(1-t)^2\right)$ are two distinct points, .: $t\ne1-t$, $t\ne\frac{1}{2}$.

Intersection of the 2 tangents:

$2tx-t^2=2(1-t)x-(1-t)^2$, $2(t-(1-t))x-\left(t^2-(1-t)^2\right)=0$

.: $2(2t-1)x-(2t-1)=0$, $(2t-1)(2x-1)=0$

Since $t\ne\frac{1}{2}$.: $x=\frac{1}{2}$, $y=t-t^2$.: $R\left(\frac{1}{2},t-t^2\right)$

Q3biv Since $R\left(\frac{1}{2},t-t^2\right)$, the locus of R is the vertical line $x=\frac{1}{2}$. $y=t-t^2=-\left(t-\frac{1}{2}\right)^2+\frac{1}{4}$ and $t\ne\frac{1}{2}$.: $y<\frac{1}{4}$

Q4ai $f(x)=e^{-x}-2e^{-2x}$, $f'(x)=-e^{-x}+4e^{-2x}$

Q4aii $f'(x)=-e^{-x}+4e^{-2x}=0$, $e^{-x}\left(-1+4e^{-x}\right)=0$

Since $e^{-x}\neq 0$.: $-1+4e^{-x}=0$, $e^{-x}=\frac{1}{4}$, $e^{x}=4$, $x=\ln 4$,

.: $y=e^{-x}-2e^{-2x}=\frac{1}{4}-2\left(\frac{1}{4}\right)^2=\frac{1}{8}$

Maximum turning point is $\left(\ln 4,\frac{1}{8}\right)$.

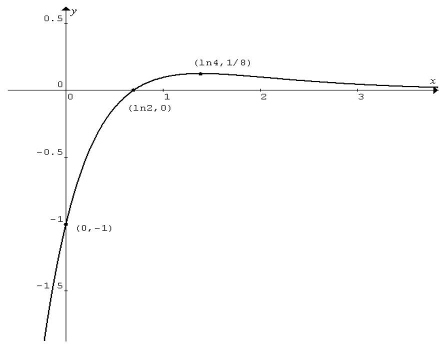
Q4aiii $f(x)=e^{-x}-2e^{-2x}$

$f(\ln 2)=e^{-\ln 2}-2e^{-2\ln 2}=\left(e^{\ln 2}\right)^{-1}-2\left(e^{\ln 4}\right)^{-1}=2^{-1}-2\times 4^{-1}=0$

Q4aiv As $x\to\infty$, $f(x)=e^{-x}\left(1-2e^{-x}\right)\to 0^{+}$

Q4av y-intercept: Let $x=0$, $y=-1$

Q4avi



Q4bi On the same arc the angle at the centre is twice the angle on the circumference.

Q4bii $\angle AOC=2x$

$\angle ADC=2x$ because the exterior angle of a triangle equals the sum of the two opposite interior angle.

.: $\angle AOC=\angle ADC$

Hence *ACDO* is a cyclic quadrilateral because $\angle AOC$ and $\angle ADC$ are on the same arc AC.

Q4biii *PM* is perpendicular to *AC* and *OM* is also perpendicular to *AC*, .: *PM* and *OM* are parallel and have a common point *M*. Hence *P*, *M* and *O* are collinear.

Q5ai $TQ=x_0=\cos\theta$, $TN=1-y_0=1-\sin\theta$, $SN=2$

$\frac{SP}{TQ}=\frac{SN}{TN}$, .: $SP=\frac{SN\times TQ}{TN}=\frac{2\cos\theta}{1-\sin\theta}$

Q5aii $\frac{\cos\theta}{1-\sin\theta}=\frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{\cos\theta(1+\sin\theta)}{\left(1-\sin^2\theta\right)}$

$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}=\frac{1+\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta$

Q5aiii $\angle SNP=\frac{1}{2}\angle SOQ=\frac{1}{2}\left(\theta+\frac{\pi}{2}\right)=\frac{\theta}{2}+\frac{\pi}{4}$

Q5aiv $\tan\angle SNP=\frac{SP}{SN}=\frac{\cos\theta}{1-\sin\theta}=\sec\theta+\tan\theta$

.: $\sec\theta+\tan\theta=\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)$

Q5av $\sec\theta+\tan\theta=\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)=\sqrt{3}$ where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, i.e.

$0<\frac{\theta}{2}+\frac{\pi}{4}<\frac{\pi}{2}$.: $\frac{\theta}{2}+\frac{\pi}{4}=\frac{\pi}{3}$, $\theta=\frac{\pi}{6}$

Q5bi $T=5+25e^{-kt}$, $20=5+25e^{-k}$, $e^{-k}=\frac{3}{5}$, $e^{k}=\frac{5}{3}$,

.: $k=\ln\frac{5}{3}$

Q5bii $T=A+Be^{-kt}$ where $A=22$ and $k=\ln\frac{5}{3}$.

Let $t=0$ at 10:00 am and $T=30$.

.: $30=22+B$, $B=8$ and $T=22+8e^{-kt}$

When $T=37$, $22+8e^{-kt}=37$, $e^{-kt}=\frac{15}{8}$, $-kt=\ln\frac{15}{8}$

$t=\frac{\ln\frac{15}{8}}{-\ln\frac{5}{3}}\approx -1.2306$, i.e. approximately 1 hour and 14 minutes before 10:00 am. The time of the day is 8:46 am approximately.

Q6a For $n=1$, $1(1+4)=\frac{1}{6}\times 1(1+1)(2\times 1+13)$ is true.

Assume $1\times 5+2\times 6+3\times 7+...+k(k+4)=\frac{1}{6}k(k+1)(2k+13)$ is true for $n=k$.

For $n=k+1$, $1\times 5+2\times 6+3\times 7+...+k(k+4)+(k+1)(k+5)$

$=\frac{1}{6}k(k+1)(2k+13)+(k+1)(k+5)$

$=\frac{1}{6}(k+1)(k(2k+13)+6(k+5))=\frac{1}{6}(k+1)\left(2k^2+19k+30\right)$

$=\frac{1}{6}(k+1)(k+2)(2k+15)=\frac{1}{6}(k+1)((k+1)+1)(2(k+1)+13)$

.: it is true for $n=k+1$

.: it is true for all $n\geq 1$

Q6bi When $y=0$, $h-\frac{1}{2}gt^2=0$, $t=\sqrt{\frac{2h}{g}}$ seconds.

Q6bii When the ball hits the ground at $45°$, the vertical and horizontal speed are the same at $t=\sqrt{\frac{2h}{g}}$.

.: $v=|\dot{y}|=gt$ and $d=vt=gt^2=2h$

Q6ci

Game 1: $\Pr(win)=\Pr(T)+\Pr(T'T)=p+(1-p)p=2p-p^2$

Q6cii Game 2: $\Pr(win)=\Pr(TT)+\Pr(T'TT)+\Pr(TT'T)$

$=p^2+(1-p)p^2+p(1-p)p=3p^2-2p^3$

Q6ciii $\Pr(winG1)-\Pr(winG2)=\left(2p-p^2\right)-\left(3p^2-2p^3\right)$

$=2p-4p^2+2p^3=2p\left(1-2p+p^2\right)=2p(1-p)^2>0$

.: $\Pr(winG1)>\Pr(winG2)$

Q6civ $\Pr(winG1)=2\times\Pr(winG2)$

$2p-p^2=2\left(3p^2-2p^3\right)$.: $4p^3-7p^2+2p=0$

$p\left(4p^2-7p+2\right)=0$

Since $0<p<1$.: $4p^2-7p+2=0$

Hence $p=\dfrac{7-\sqrt{49-32}}{8}=\dfrac{7-\sqrt{17}}{8}$.

Q7ai The semi-vertical angle is $45°$, .: radius = height.

Volume remaining $V=\dfrac{\pi}{3}h^3-\dfrac{\pi}{3}\ell^3=\dfrac{\pi}{3}\left(h^3-\ell^3\right)$

Q7aii $\dfrac{dV}{dt}=\dfrac{dV}{d\ell}\times\dfrac{d\ell}{dt}=-\pi\ell^2\times\dfrac{d\ell}{dt}$

When $\ell=2$, $\dfrac{dV}{dt}=-\pi\left(2^2\right)(10)=-40\pi$.

Volume is decreasing at 40π cm^3 per second.

Q7aiii When $\dfrac{\pi}{3}\ell^3=\dfrac{1}{8}\times\dfrac{\pi}{3}h^3$, $\ell=\dfrac{h}{2}$

$\dfrac{dV}{dt}=-\pi\ell^2\times\dfrac{d\ell}{dt}=-\pi\left(\dfrac{h^2}{4}\right)(10)=-\dfrac{5}{2}\pi h^2$ cm^3 per second.

Q7bi $\displaystyle\sum_{r=0}^{n}\binom{n}{r}x^r=(1+x)^n$ for $n\geq 1$.

$\displaystyle\frac{d}{dx}\sum_{r=0}^{n}\binom{n}{r}x^r=\frac{d}{dx}(1+x)^n$, $\displaystyle\sum_{r=1}^{n}\binom{n}{r}rx^{r-1}=n(1+x)^{n-1}$

$\displaystyle x\times\sum_{r=1}^{n}\binom{n}{r}rx^{r-1}=x\times n(1+x)^{n-1}$

.: $\displaystyle\sum_{r=1}^{n}\binom{n}{r}rx^r=nx(1+x)^{n-1}$

Q7bii $\displaystyle\frac{d}{dx}\sum_{r=1}^{n}\binom{n}{r}rx^r=\frac{d}{dx}nx(1+x)^{n-1}$

$\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2x^{r-1}=n(n-1)x(1+x)^{n-2}+n(1+x)^{n-1}$

$\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2x^{r-1}=n(1+x)^{n-2}(nx+1)$

Let $x=1$, $\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2=n(n+1)2^{n-2}$

Q7biii From part (ii), $\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2x^{r-1}=n(1+x)^{n-2}(nx+1)$

Let $x=-1$, $\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2(-1)^{r-1}=0$

.: $\binom{n}{1}1^2+\binom{n}{3}3^2+\ldots+\binom{n}{n-1}(n-1)^2-\binom{n}{2}2^2-\binom{n}{4}4^2-\ldots-\binom{n}{n}n^2=0$

.: $\binom{n}{1}1^2+\binom{n}{3}3^2+\ldots+\binom{n}{n-1}(n-1)^2=\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2$

$\displaystyle\sum_{r=1}^{n}\binom{n}{r}r^2=n(n+1)2^{n-2}$

$\binom{n}{1}1^2+\binom{n}{3}3^2+\ldots+\binom{n}{n-1}(n-1)^2+\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2=n(n+1)2^{n-2}$

.: $2\left(\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2\right)=n(n+1)2^{n-2}$

.: $\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2=\dfrac{n(n+1)2^{n-2}}{2}=n(n+1)2^{n-3}$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.