2011 NSW BOS Mathematics Extension 2 Solutions

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Q1a Let $u = \ln x$ and $\frac{dv}{dx} = x$, .: $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$

$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{x^2}{2} \times \frac{1}{x} dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

Q1b Let $u = x+1$, .: $x = u-1$, $\frac{du}{dx} = 1$

$\int_0^3 x\sqrt{x+1}dx = \int_0^3 (u-1)\sqrt{u}\frac{du}{dx}dx = \int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

$= \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_1^4 = \frac{64}{5} - \frac{16}{3} - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{116}{15}$

Q1ci $\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} = \frac{1}{x^2(x-1)}$,

$\frac{ax(x-1) + b(x-1) + cx^2}{x^2(x-1)} = \frac{1}{x^2(x-1)}$,

.: $ax(x-1) + b(x-1) + cx^2 = 1$

.: $b = -1$, $a + c = 0$, $b - a = 0$, .: $a = -1$ and $c = 1$

Q1cii $\int \frac{1}{x^2(x-1)}dx = \int \left(\frac{-1}{x} + \frac{-1}{x^2} + \frac{1}{x-1}\right)dx$

$= -\ln|x| + \frac{1}{x} + \ln|x-1| + c = \ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + c$

Q1d $\int \cos^3 \theta d\theta = \int \cos^2 \theta \cos \theta d\theta = \int (1 - \sin^2 \theta)\cos \theta d\theta$

$= \int (1 - u^2)\frac{du}{d\theta} d\theta$ (where $u = \sin \theta$ and $\frac{du}{d\theta} = \cos \theta$)

$= u - \frac{1}{3}u^3 + c = \sin \theta - \frac{1}{3}\sin^3 \theta + c$

Q1e $\int_{-1}^1 \frac{1}{5 - 2t + t^2}dt = \int_{-1}^1 \frac{1}{4 + t^2 - 2t + 1}dt = \int_{-1}^1 \frac{1}{4 + (t-1)^2}dt$

$= \frac{1}{2}\int_{-1}^1 \frac{2}{4 + (t-1)^2}dt = \left[\frac{1}{2}\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^1 = \left(0 - \frac{1}{2}\tan^{-1}(-1)\right) = \frac{\pi}{8}$

Q2ai $\overline{w} + z = (2+3i) + (3+4i) = 5 + 7i$

Q2aii $|w| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

Q2aiii $\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{-6-17i}{25} = -\frac{6}{25} - \frac{17}{25}i$

Q2bi $z = (1 + \sqrt{3}i) + (\sqrt{3} + i) = (1 + \sqrt{3}) + (1 + \sqrt{3})i$

Q2bii $Arg(\sqrt{3} + i) = \frac{\pi}{6}$, $Arg(1 + \sqrt{3}i) = \frac{\pi}{3}$

$\theta = \pi - \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{5\pi}{6}$

Q2c $z^3 = 8$, one obvious solution is $z = 2$, the other two solutions lie on the circle of radius 2 centred at the origin. The three solutions are separated from each other by $\frac{2\pi}{3}$.

The solutions: $z = 2 = 2cis0$, $2cis\left(-\frac{2\pi}{3}\right)$, $2cis\left(\frac{2\pi}{3}\right)$.

Q2di $(\cos\theta + i\sin\theta)^3$

$= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Q2dii $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (de Moivre's theorem)

.: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (real parts)

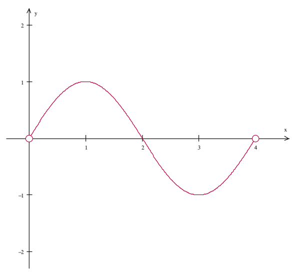
$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta$

.: $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

Q2diii $4\cos^3\theta - 3\cos\theta = 1$, $\cos 3\theta = 1$, $3\theta = 2k\pi$,

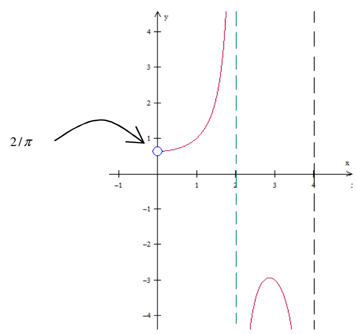
$\theta = \frac{2\pi}{3}$ is the smallest positive value when $k = 1$.

Q3ai $y = \sin\frac{\pi}{2}x$, $0 < x < 4$



Q3aii $\lim_{x\to 0}\frac{x}{\sin\frac{\pi}{2}x} = \frac{2}{\pi}\lim_{x\to 0}\frac{\frac{\pi}{2}x}{\sin\frac{\pi}{2}x} = \frac{2}{\pi}$

Q3aiii



Q3b

Base of isosceles triangle $= 2\cos x$

Height $= \sqrt{1-\cos^2 x} = \sin x$

Area $= \frac{1}{2}\times 2\cos x \times \sin x = \frac{1}{2}\sin 2x$

Volume of solid $= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2x dx = \left[-\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$ cubic units

Q3c To prove $(2n)! \ge 2^n (n!)^2$ for all positive integers n.

For $n=1$, $2! \ge 2^1(1!)^2$ is true.

Assume $(2k)! \ge 2^k (k!)^2$, i.e. $\frac{(2k)!}{2^k (k!)^2} \ge 1$

For $n = k+1$,

$$\frac{(2n)!}{2^n (n!)^2} = \frac{(2k+2)!}{2^{k+1}(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{2\times 2^k (k+1)^2 (k!)^2}$$

$$= \frac{2(k+1)(2k+1)(2k)!}{2\times 2^k (k+1)^2 (k!)^2} = \frac{(2k+1)(2k)!}{(k+1)2^k (k!)^2} \ge 1 \text{ is true.}$$

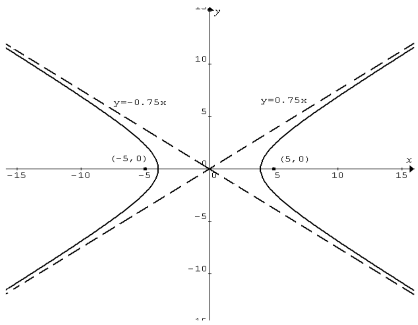
.: $(2n)! \ge 2^n (n!)^2$ is true for all positive integers n.

Q3di $b^2 = a^2(e^2-1)$, $9 = 16(e^2-1)$, $e = \frac{5}{4}$

Q3dii Foci are $(\pm ae, 0) = (\pm 5, 0)$

Q3diii $y = \pm\frac{b}{a}x$, $y = \pm\frac{3}{4}x$

Q3div



Q3dv $b^2 = a^2(e^2-1)$, $\frac{b}{a} = \sqrt{e^2-1}$, as $e \to \infty$, $\frac{b}{a} \to \infty$, the two asymptotes become vertical and the same line as the y-axis, also the two branches of the general hyperbola become a vertical line along the y-axis.

Q4ai $|z-a|^2 - |z-b|^2 = 1$, $|x-a+iy|^2 - |x-b+iy|^2 = 1$

$(x-a)^2 + y^2 - (x-b)^2 - y^2 = 1$, $-2ax + a^2 + 2bx - b^2 = 1$

$2(b-a)x = b^2 - a^2 + 1$, $x = \frac{(b-a)(b+a)+1}{2(b-a)} = \frac{a+b}{2} + \frac{1}{2(b-a)}$

Q4aii The locus of z is the vertical line through the x-axis at $x = \frac{a+b}{2} + \frac{1}{2(b-a)}$.

Q4bi $\angle ABC + \angle ADC = 180°$ (opposite angles of a cyclic quadrilateral)

$\angle ADG + \angle ADC = 180°$ (supplementary angles)

$\angle ABC + \angle AFG = 180°$ (co-interior angles)

.: $\angle ADG + \angle AFG = 180°$, .: *FADG* is cyclic.

Q4bii Alternate angles are equal.

Q4biii *FADG* is cyclic, .: $\angle GFD = \angle GAD$

From part ii, $\angle GFD = \angle AED$, .: $\angle GAD = \angle AED$

.: *GA* is a tangent to the circle *ABCD* because the angle between *GA* (tangent) and *AD* (chord) equals the angle in the alternate segment.

Q4ci $y = Af(t) + Bg(t)$, $\frac{dy}{dt} = Af'(t) + Bg'(t)$

$\frac{d^2y}{dt^2} = Af''(t) + Bg''(t)$

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y$

$= A(f''(t) + 3f'(t) + 2f(t)) + B(g''(t) + 3g'(t) + 2g(t))$

$= A\times 0 + B\times 0 = 0$

.: $y = Af(t) + Bg(t)$ is also a solution.

Q4aii $y = e^{kt}$, $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$, $k^2e^{kt} + 3ke^{kt} + 2e^{kt} = 0$

$(k^2 + 3k + 2)e^{kt} = 0$, $(k+2)(k+1)e^{kt} = 0$.

Since $e^{kt} > 0$, .: $k = -2$ or -1.

Q4ciii $y = Ae^{-2t} + Be^{-t}$, $\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t}$

When $t = 0$, $y = 0$, .: $A + B = 0$ and

$\frac{dy}{dt} = 1$, .: $-2A - B = 1$

.: $A = -1$ and $B = 1$

Q5ai $a = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$

Horizontal component of resultant force $= ma$

$F\sin\theta - N\sin\theta = m\omega^2 r$

Vertical component of resultant force $= 0$

$F\cos\theta + N\cos\theta - mg = 0$, .: $F\cos\theta + N\cos\theta = mg$

Q5aii $F - N = \frac{m\omega^2 r}{\sin\theta}$, $F + N = \frac{mg}{\cos\theta}$

.: $2N = \frac{mg}{\cos\theta} - \frac{m\omega^2 r}{\sin\theta}$, .: $N = \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r \operatorname{cosec}\theta$

Q5aiii $\cos\theta = \frac{h}{R}$, $\sin\theta = \frac{r}{R}$

To remain in contact, $N \geq 0$.

$\therefore\ \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r \operatorname{cosec}\theta \geq 0$

$\therefore\ \frac{1}{2}mg\frac{R}{h} - \frac{1}{2}m\omega^2 r\frac{R}{r} \geq 0$, $\frac{g}{h} - \omega^2 \geq 0$, $\omega^2 \leq \frac{g}{h}$, $\therefore\ \omega \leq \sqrt{\frac{g}{h}}$.

Q5b $\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r}$

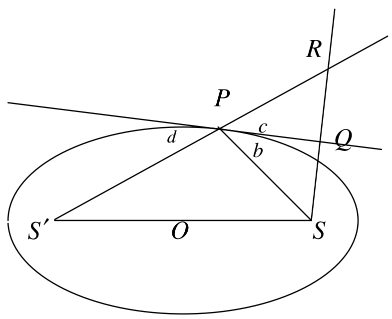
$= \frac{p(1+q)(1+r) + q(1+r)(1+p) - r(1+p)(1+q)}{(1+p)(1+q)(1+r)}$

$= \frac{(p+q-r) + 2pq + pqr}{(1+p)(1+q)(1+r)}$

Since p, q, r are positive real numbers and $p+q \geq r$, $\therefore$ the numerator and denominator are both positive,

$\therefore\ \frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0$

Q5ci



$\angle d = \angle b$ (reflective property), $\angle d = \angle c$ (vertically opposite angles), $\therefore\ \angle b = \angle c$, $\angle PQR = \angle PQS = 90°$, PQ is a common side, $\therefore\ \Delta PRQ$ and ΔPSQ are congruent and $SQ = RQ$

Q5cii ΔPRQ and ΔPSQ are congruent, $\therefore\ SP = PR$

Since $S'P + SP = 2a$, $\therefore\ S'P + PR = S'R = 2a$

Q5ciii O is the midpoint of SS' and Q is the midpoint of SR, $\therefore\ OQ = \frac{1}{2}S'R = a$, $\therefore$ Q is at a constant distance a from the origin O, i.e. it lies on the circle $x^2 + y^2 = a^2$.

Q6ai $m\frac{dv}{dt} = mg - kv^2$, when $v = v_T$, $\frac{dv}{dt} = 0$

$mg - k{v_T}^2 = 0$, $v_T = \sqrt{\frac{mg}{k}}$

Q6aii $\frac{dv}{dt} = \frac{mg - kv^2}{m}$, $\frac{dt}{dv} = \frac{1}{g}\frac{\frac{mg}{k}}{\frac{mg}{k} - v^2} = \frac{{v_T}^2}{g}\frac{1}{{v_T}^2 - v^2}$

$t = \frac{{v_T}^2}{g}\int \frac{1}{{v_T}^2 - v^2}\,dv = \frac{v_T}{2g}\int\left(\frac{1}{v_T - v} + \frac{1}{v_T + v}\right)dv$

$= \frac{v_T}{2g}\left(-\ln(v_T - v) + \ln(v_T + v) + c\right)$

$= \frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v} + c\right)$

At $t = 0$, $v = v_0$, $\frac{v_T}{2g}\left(\ln\frac{v_T + v_0}{v_T - v_0} + c\right) = 0$, $\therefore\ c = -\ln\frac{v_T + v_0}{v_T - v_0}$

$\therefore\ t = \frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v} - \ln\frac{v_T + v_0}{v_T - v_0}\right) = \frac{v_T}{2g}\left(\ln\frac{(v_T + v)(v_T - v_0)}{(v_T - v)(v_T + v_0)}\right)$

Q6aiii Jac: $\Delta t = \left[\frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v}\right)\right]_{\frac{1}{3}v_T}^{\frac{2}{3}v_T} = \frac{v_T}{2g}\ln\frac{5}{2}$

Gil: $\Delta t = \left[\frac{v_T}{2g}\left(\ln\left|\frac{v_T + v}{v_T - v}\right|\right)\right]_{3v_T}^{\frac{3}{2}v_T} = \frac{v_T}{2g}\ln\frac{5}{2}$

$\therefore$ the same time for Jac's speed to double and Gil's speed to halve.

Q6bi $y = (f(x))^3$, $\frac{dy}{dx} = 3(f(x))^2 f'(x)$

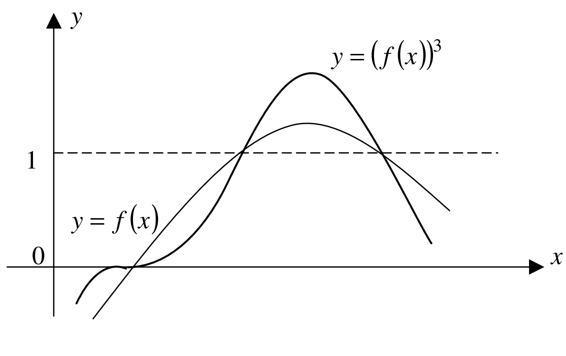
At $x = a$, $\frac{dy}{dx} = 3(f(a))^2 f'(a)$

If $f(a) = 0$ or $f'(a) = 0$, $\frac{dy}{dx} = 0$, $\therefore\ y = (f(x))^3$ has a stationary point at $x = a$.

Q6bii If $f(a) = 0$, $(f(a))^3 = 0$, $\therefore$ both $y = f(x)$ and $y = (f(x))^3$ have an x-intercept at $x = a$.

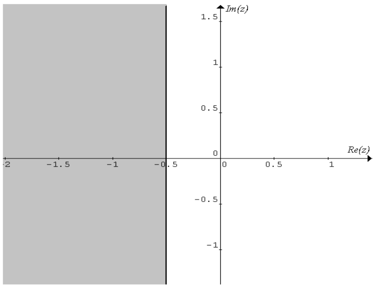
For $y = f(x)$, $(a,0)$ is not a stationary point. For $y = (f(x))^3$, it is a stationary point and a triple root, $\therefore\ (a,0)$ is a horizontal point of inflexion of $y = (f(x))^3$.

Q6biii



Q6c $\left|1+\frac{1}{z}\right|\le 1$, $\left|\frac{z+1}{z}\right|\le 1$, $\frac{|z+1|}{|z|}\le 1$, $|z+1|\le|z|$

$|z+1|\le|z|$ defines a region where the distance of z from $z=-1$ is less than or equal to its distance from $z=0$. The region is to the left of the line (including the line) $\text{Re}(z)=-\frac{1}{2}$.



Q7a *Volume of a cylindrical shell* $=2\pi rh dx=2\pi(1-x)ydx$

$=2\pi(1-x)\frac{x}{1+x^2}dx=2\pi\left(\frac{x}{1+x^2}-1+\frac{1}{1+x^2}\right)dx$

Volume of solid $=\int_0^1 2\pi\left(\frac{x}{1+x^2}-1+\frac{1}{1+x^2}\right)dx$

$=2\pi\left[\frac{1}{2}\ln\left(1+x^2\right)-x+\tan^{-1}x\right]_0^1$

$=2\pi\left(\frac{1}{2}\ln 2-1+\frac{\pi}{4}\right)$

Q7bi $I=\int_1^3\frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx$. Let $u=4-x$.

$I=-\int_3^1\frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{u(4-u)}du=\int_1^3\frac{\cos^2\left(\frac{\pi}{2}-\frac{\pi}{8}u\right)}{u(4-u)}du=\int_1^3\frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}du$

Q7bii $2I=\int_1^3\frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx+\int_1^3\frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}du$

$2I=\int_1^3\frac{\cos^2\left(\frac{\pi}{8}x\right)+\sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx=\int_1^3\frac{1}{x(4-x)}dx$

$=\frac{1}{4}\int_1^3\left(\frac{1}{x}+\frac{1}{4-x}\right)dx=\frac{1}{4}\left[\ln\frac{x}{4-x}\right]_1^3=\frac{1}{4}\left(\ln 3-\ln\frac{1}{3}\right)=\frac{1}{2}\ln 3$

$\therefore\ I=\frac{1}{4}\ln 3$

Q7ci $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y=mx+c$, $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$b^2x^2+a^2(mx+c)^2-a^2b^2=0$

$\left(b^2+a^2m^2\right)x^2+\left(2a^2mc\right)x+a^2\left(c^2-b^2\right)=0$

Only one point of intersection, $\Delta=0$,

$\left(2a^2mc\right)^2-4\left(b^2+a^2m^2\right)a^2\left(c^2-b^2\right)=0$

$4a^4m^2c^2-4a^2b^2c^2-4a^4m^2c^2+4a^2b^4+4a^4b^2m^2=0$

$\therefore\ a^2m^2+b^2=c^2$

Q7cii y-intercept of ℓ is $C(0,c)$, x-intercept of ℓ is $R\left(-\frac{c}{m},0\right)$

$RC=\sqrt{\frac{c^2}{m^2}+c^2}$, $RS=\left|-\frac{c}{m}-ae\right|=\left|\frac{c}{m}+ae\right|$

ΔCOR and ΔSQR are similar, $\therefore\ \frac{QS}{OC}=\frac{RS}{RC}$

$\therefore\ \frac{QS}{|c|}=\frac{\left|\frac{c}{m}+ae\right|}{\sqrt{\frac{c^2\left(1+m^2\right)}{m^2}}}=\frac{\left|\frac{c}{m}+ae\right|}{\frac{|c|}{|m|}\sqrt{1+m^2}}$, $\therefore\ QS=\frac{|mae+c|}{\sqrt{1+m^2}}$

Q7ciii $QS=\frac{|mae+c|}{\sqrt{1+m^2}}$ and given $Q'S'=\frac{|mae-c|}{\sqrt{1+m^2}}$,

$QS\times Q'S'=\frac{|mae+c|}{\sqrt{1+m^2}}\times\frac{|mae-c|}{\sqrt{1+m^2}}=\frac{\left|m^2a^2e^2-c^2\right|}{1+m^2}$

$=\frac{\left|m^2a^2e^2-a^2m^2-b^2\right|}{1+m^2}=\frac{\left|-m^2a^2\left(1-e^2\right)-b^2\right|}{1+m^2}=\frac{\left|-m^2b^2-b^2\right|}{1+m^2}$

$=\frac{b^2\left(1+m^2\right)}{1+m^2}=b^2$

Q8a $I_m=\int_0^1 x^m\left(x^2-1\right)^5dx=\int_0^1 x^{m-1}x\left(x^2-1\right)^5dx$

Integration by parts: $\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

Let $u=x^{m-1}$, $\frac{du}{dx}=(m-1)x^{m-2}$

Let $\frac{dv}{dx}=x\left(x^2-1\right)^5$, $v=\frac{1}{12}\left(x^2-1\right)^6$

$\therefore\ I_m=\left[\frac{1}{12}x^{m-1}\left(x^2-1\right)^6\right]_0^1-\int_0^1\frac{1}{12}\left(x^2-1\right)^6(m-1)x^{m-2}dx$

$I_m=-\frac{m-1}{12}\int_0^1\left(x^2-1\right)^6x^{m-2}dx=-\frac{m-1}{12}\int_0^1\left(x^2-1\right)^5\left(x^m-x^{m-2}\right)dx$

$\therefore\ I_m=-\frac{m-1}{12}\left(\int_0^1 x^m\left(x^2-1\right)^5dx-\int_0^1 x^{m-2}\left(x^2-1\right)^5dx\right)$

$I_m=-\frac{m-1}{12}\left(I_m-I_{m-2}\right)$, $-12I_m=mI_m-I_m-(m-1)I_{m-2}$

$-(m+11)I_m=-(m-1)I_{m-2}$, $\therefore\ I_m=\frac{m-1}{m+11}I_{m-2}$

Q8bi Pr*(each ball is selected exactly once)* $= \frac{7!}{7^7} = \frac{6!}{7^6}$

Q8bii Pr*(at least one ball is not selected)*
$= 1 -$ Pr*(none is not selected)*
$= 1 -$ Pr*(each ball is selected exactly once)*
$= 1 - \frac{6!}{7^6}$

Q8biii There are ${}^7C_6$ ways to select 6 different balls out of 7, and ${}^6C_1$ ways to select the remaining ball out of the 6 already selected balls. The 7 balls can be arranged in $\frac{7!}{2!}$ ways.

.: Pr*(exactly one of the balls is not selected)*

$$= \frac{{}^7C_6 \times {}^6C_1 \times \frac{7!}{2!}}{7^7} = \frac{3 \times 6!}{7^5}$$

Q8ci β is a root of $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} ... + a_1 z + a_0$

.: $P(\beta) = \beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} ... + a_1\beta + a_0 = 0$

.: $\beta^n = -a_{n-1}\beta^{n-1} - a_{n-2}\beta^{n-2} - ... - a_1\beta - a_0$

$|\beta^n| = |a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + ... + a_1\beta + a_0|$

$\le |a_{n-1}\beta^{n-1}| + |a_{n-2}\beta^{n-2}| + ... + |a_1\beta| + |a_0|$

$= |a_{n-1}||\beta^{n-1}| + |a_{n-2}||\beta^{n-2}| + ... + |a_1||\beta| + |a_0|$

$\le M|\beta^{n-1}| + M|\beta^{n-2}| + ... + M|\beta| + M$

.: $|\beta^n| \le M(|\beta^{n-1}| + |\beta^{n-2}| + ... + |\beta| + 1)$

.: $|\beta|^n \le M(|\beta|^{n-1} + |\beta|^{n-2} + ... + |\beta| + 1)$

Q8cii M is the maximum value of $|a_{n-1}|$, $|a_{n-2}|$, …, $|a_0|$.

If $0 < |\beta| < 1$, $|\beta| < 1 + M$ since $M > 0$.

If $|\beta| > 1$, $|\beta|^n - 1 = (|\beta| - 1)(|\beta|^{n-1} + |\beta|^{n-2} + ... + |\beta| + 1)$

.: $(|\beta|^{n-1} + |\beta|^{n-2} + ... + |\beta| + 1) = \frac{(|\beta|^n - 1)}{(|\beta| - 1)}$

.: $|\beta|^n \le M(|\beta|^{n-1} + |\beta|^{n-2} + ... + |\beta| + 1) = M\frac{(|\beta|^n - 1)}{(|\beta| - 1)} < M\frac{|\beta|^n}{|\beta| - 1}$

.: $1 < \frac{M}{|\beta| - 1}$, $|\beta| - 1 < M$, .: $|\beta| < 1 + M$

.: for any root β of $P(z)$, $|\beta| < 1 + M$

Q8d $S(x) = \sum_{k=0}^{n} c_k\left(x + \frac{1}{x}\right)^k$, $\frac{S(x)}{c_n} = \sum_{k=0}^{n} \frac{c_k}{c_n}\left(x + \frac{1}{x}\right)^k$ which is in the form of $P(z)$.

Let M be the maximum value of $\left|\frac{c_k}{c_n}\right|$.

Given $|c_k| \le |c_n|$, .: $\left|\frac{c_k}{c_n}\right| \le 1$, .: $M \le 1$.

For any root $\beta = \alpha + \frac{1}{\alpha}$ of $\frac{S(x)}{c_n} = 0$, $|\beta| < 1 + M$

.: $\left|\alpha + \frac{1}{\alpha}\right| < 1 + M \le 2$

.: $\left(\frac{\alpha^2 + 1}{\alpha}\right)^2 < 4$, $(\alpha^2 + 1)^2 < 4\alpha^2$, $(\alpha^2 + 1)^2 - 4\alpha^2 < 0$

.: $(\alpha^2 - 1)^2 < 0$

No real α can satisfy the last inequality, .: $\frac{S(x)}{c_n} = 0$ and hence $S(x) = 0$ has no real solutions.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.