2011 NSW BOS Mathematics Exam Solutions

Q1a $\sqrt[3]{\frac{651}{(4\pi)}} \approx 3.728$ by calculator.

Q1b $\frac{n^2-25}{n-5} = \frac{(n-5)(n+5)}{n-5} = n+5$

Q1c $2^{2x+1} = 32$, $2^{2x+1} = 2^5$, $2x+1=5$, $x=2$

Q1d $\frac{d}{dx}(\ln(5x+2)) = \frac{5}{5x+2}$

Q1e $2-3x \le 8$, $-3x \le 6$, $x \ge \frac{6}{-3}$, $x \ge -2$

Q1f

$$\frac{4}{\sqrt{5}-\sqrt{3}} = \frac{4}{\left(\sqrt{5}-\sqrt{3}\right)} \times \frac{\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)} = \frac{4\left(\sqrt{5}+\sqrt{3}\right)}{2} = 2\left(\sqrt{5}+\sqrt{3}\right)$$

Q1g $800 \times 0.02 = 16$ defective items

Q2ai $\alpha + \beta = 6$

Q2aii $\alpha\beta = 2$

Q2aiii $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{6}{2} = 3$

Q2b $2\sin x = -\sqrt{3}$ and $0 \le x \le 2\pi$, $\sin x = -\frac{\sqrt{3}}{2}$

$x = \frac{4\pi}{3}, \frac{5\pi}{3}$

Q2c $y = (2x+1)^4$, $\frac{dy}{dx} = 8(2x+1)^3$

At $x=-1$, $y=1$, $\frac{dy}{dx} = -8$

Tangent: $y-1 = -8(x+1)$, $y = -8x-7$

Q2d $y = x^2e^x$, $\frac{dy}{dx} = 2xe^x + x^2e^x$ by the product rule

Q2e $\int \frac{1}{3x^2}dx = -\frac{1}{3x} + c$

Q3ai $t_1 = 3$, $d = 0.5$, $t_{25} = 3 + 24 \times 0.5 = 15$, \$15 million

Q3aii $S_{110} = \frac{110}{2}(2\times 3 + 109 \times 0.5) = 3327.5$, \$3327.5 million

Q3b The vertex of the parabola is equidistant from its focus $(3,2)$ and directrix $y=-4$, .: the vertex is $(3,-1)$.

Q3ci ℓ_1: $3x+4y-12=0$

Let $x=0$, $y=3$, .: $B(0,3)$

Q3cii $m_1 = -\frac{3}{4}$, $m_2 = \frac{4}{3}$, $m_1m_2 = -1$, .: ℓ_1 and ℓ_2 are $\perp$

Q3ciii ℓ_1: $3x+4y-12=0$. Let $y=0$, $x=4$, .: $A(4,0)$

.: $AB = \sqrt{3^2+4^2} = 5$

ΔOAB and ΔOEB, .: $\frac{OE}{OA} = \frac{OB}{AB}$, $\frac{OE}{4} = \frac{3}{5}$, $OE = \frac{12}{5}$

Q3civ $\frac{BE}{BO} = \frac{OE}{OA}$, $\frac{BE}{3} = \frac{\frac{12}{5}}{4}$, $BE = \frac{9}{5}$

Q3cv $\textit{Area of } \Delta BOE = \frac{1}{2} \times \frac{12}{5} \times \frac{9}{5} = \frac{54}{25}$ unit squares

Q4a $\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{\sin x - x\cos x}{\sin^2 x}$

Q4b $\int_e^{e^3} \frac{5}{x}dx = [5\ln x]_e^{e^3} = 5\ln e^3 - 5\ln e = 15 - 5 = 10$

Q4c $\frac{dy}{dx} = 6x-2$, $y = \int (6x-2)dx = 3x^2 - 2x + c$

$(-1,4)$, $4 = 3(-1)^2 - 2(-1) + c$, $c = -1$

.: $y = 3x^2 - 2x - 1$

Q4di $y = \sqrt{9-x^2}$, $\frac{dy}{dx} = \frac{1}{2\sqrt{9-x^2}} \times (-2x) = -\frac{x}{\sqrt{9-x^2}}$

Q4dii $-\frac{x}{\sqrt{9-x^2}} = \frac{dy}{dx}$, $\frac{6x}{\sqrt{9-x^2}} = -6\frac{dy}{dx}$

$$\int \frac{6x}{\sqrt{9-x^2}}dx = -6\int \frac{dy}{dx}dx = -6\int dy = -6y + c = -6\sqrt{9-x^2} + c$$

Q4e $y \le 4 - x^2$ and $y \ge |x| - 2$

Q5ai $t_1 = a = 27$, $r = 2$, $t_{12} = ar^{12-1} = 27\times 2^{11} = 55296$

Q5aii $t_n = 27\times 2^{n-1} > 10000000$, $2^{n-1} > \frac{10000000}{27}$,
$n > 19.5$, .: $n = 20$, i.e. Day 20.

Q5aiii $S_{12} = \frac{27(2^{12}-1)}{2-1} = 110565$,
$earnings = \$0.005\times 110565 \approx \$553$

Q5bi $\Pr(red\_shirt\_Monday) = \frac{3}{5}$

Q5bii $\Pr(red\_red\_red) = \frac{3}{5}\times\frac{2}{4}\times\frac{1}{3} = \frac{1}{10}$

Q5biii $\Pr(red\_yellow\_red) + \Pr(yellow\_red\_yellow)$
$= \frac{3}{5}\times\frac{2}{4}\times\frac{2}{3} + \frac{2}{5}\times\frac{3}{4}\times\frac{1}{3} = \frac{3}{10}$

Q5c $\int_0^{20} v\,dt \approx \frac{5}{3}(173 + 4\times 81 + 2\times 127 + 4\times 195 + 168) = \frac{8495}{3}$ m

Q6ai $\angle CDE = \frac{3\times 180°}{5} = 108°$

Q6aii $\angle CED = \frac{180° - 108°}{2} = 36°$
$\angle EPC = 360° - 3\times 108° = 36°$
.: ΔEPC is isosceles

Q6b $\overline{PA}^2 + \overline{PB}^2 = 40$, $(x+1)^2 + y^2 + (x-3)^2 + y^2 = 40$,
$2x^2 - 4x + 10 + 2y^2 = 40$, $x^2 - 2x + 5 + y^2 = 20$,
$x^2 - 2x + 1 + y^2 = 16$, $(x-1)^2 + y^2 = 4^2$
.: the locus is a circle of radius 4 centred at $(1,0)$.

Q6ci $P(0,2)$

Q6cii $\int_0^{\frac{\pi}{2}} 2\cos x\,dx = [2\sin x]_0^{\frac{\pi}{2}} = 2\sin\frac{\pi}{2} - 2\sin 0 = 2$

Q6ciii C

Q6civ $Area = 2\times 4 = 8$ unit squares

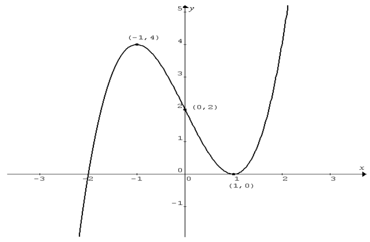
Q6cv $\int_{\frac{\pi}{2}}^{2\pi} 2\cos x\,dx = -2 - 2 + 2 = -2$

Q7ai $f(x) = x^3 - 3x + 2$, $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$
$f'(x) = 0$ at $x = -1$, $y = 4$ or $x = 1$, $y = 0$
Stationary points are $(-1,4)$ and $(1,0)$
Since the coefficient of the x^3 term is a positive value, the shape of $f(x)$ is  .
OR

x	-2	-1	0	1	2
$f'(x)$	+	0	–	0	+

.: $(-1,4)$ is a local maximum and $(1,0)$ is a local minimum.

Q7aii



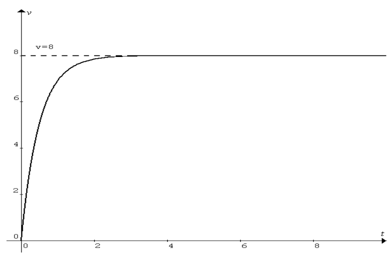
Q7bi $\dot{x} = 8 - 8e^{-2t}$
When $t = 0$, $v = \dot{x} = 8 - 8e^0 = 0$, .: the particle is initially at rest.

Q7bii $a = \ddot{x} = 16e^{-2t} > 0$, .: the acceleration is always positive.

Q7biii The particle starts from rest and its acceleration is always positive, .: its velocity is positive for all $t > 0$, i.e. the particle moves in the positive direction for all $t > 0$.

Q7biv $v = \dot{x} = 8 - 8e^{-2t} \to 8$ as $t \to \infty$. The value of the constant is 8.

Q7bv



Q8ai $x^2 + 20^2 - 2(x)(20)\cos 60° = 22^2$
$x^2 + 400 - 20x = 484$, .: $x^2 - 20x - 84 = 0$

Q8aii $x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(1)(-84)}}{2} = \frac{20 \pm \sqrt{736}}{2}$
.: $PL = \frac{20 + \sqrt{736}}{2} \approx 24$ km

Q8bi $V = \int_0^h \pi x^2\,dy = \int_0^h \pi y\,dy = \left[\frac{\pi y^2}{2}\right]_0^h = \frac{1}{2}\pi h^2$

Q8bii $V_{cylinder} = \pi\left(\sqrt{h}\right)^2 h = \pi h^2$

$\therefore \dfrac{V_{paraboloid}}{V_{cylinder}} = \dfrac{\frac{1}{2}\pi h^2}{\pi h^2} = \dfrac{1}{2}$

Q8ci $R = 1 + \dfrac{6}{12 \times 100} = 1.005$

End of *n*th month	$P_n
1	100×1.005^1
2	$100 \times 1.005^1 + 100 \times 1.005^2$
3	$100 \times 1.005^1 + 100 \times 1.005^2 + 100 \times 1.005^3$
…	…
420	$\dfrac{100 \times 1.005\left(1.005^{420} - 1\right)}{0.005} \approx 143183$

Q8cii1

End of *n*th month	$A_n
1	$29227 \times 1.005 + M \times 1.005$
2	$\left(29227 \times 1.005 + M + M \times 1.005\right) \times 1.005$ $= 29227 \times 1.005^2 + M\left(1.005 + 1.005^2\right)$

Q8cii2 At the end of 240th month after starting the new contributions, A_{240}

$= 29227 \times 1.005^{240} + M\left(1.005 + 1.005^2 + ... + 1.005^{240}\right) = 800000$

$\therefore M = \dfrac{800000 - 29227 \times 1.005^{240}}{1.005 + 1.005^2 + ... + 1.005^{240}} = \dfrac{703252.6538}{464.3511} = 1514.48$

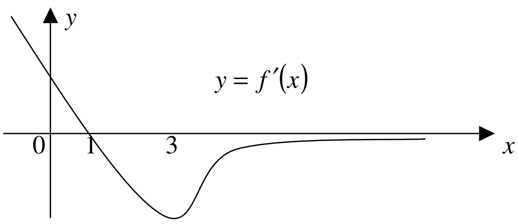
Q9ai ΔABC and ΔADE are similar because $\dfrac{AB}{AD} = \dfrac{AC}{AE} = \dfrac{BC}{DE} = \dfrac{1}{2}$.

Q9aii BC and DE are parallel, $\therefore \angle BCD = \angle CDE$ and $\angle CBE = \angle BED$, $\therefore \Delta FBC$ and ΔFED are similar.
Since $BC : DE = 1:2$, $\therefore BF : FE = 1:2$

Q9bi $\left(2 + \dfrac{t^2}{t+1}\right) - \left(1 + \dfrac{1}{t+1}\right) = 1 + \dfrac{t^2 - 1}{t+1} = \dfrac{t^2 + t}{t+1} = \dfrac{t(t+1)}{t+1} = t$

Q9bii $\Delta V = \displaystyle\int_0^4 t\,dt = \left[\dfrac{t^2}{2}\right]_0^4 = 8$ litres

Q9c



Q9di $\dfrac{1}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \times \dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$

$= \dfrac{\sqrt{n+1} - \sqrt{n}}{n+1-n} = \sqrt{n+1} - \sqrt{n}$

Q9dii $\dfrac{1}{\sqrt{1} + \sqrt{2}} + \dfrac{1}{\sqrt{2} + \sqrt{3}} + \dfrac{1}{\sqrt{3} + \sqrt{4}} + ... + \dfrac{1}{\sqrt{99} + \sqrt{100}}$

$= \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + ... + \sqrt{100} - \sqrt{99}$

$= \sqrt{100} - \sqrt{1} = 9$

Q10ai $I = 10^{-12} \times e^{0.1L} = 10^{-12} \times e^{0.1 \times 110} \approx 6.0 \times 10^{-8}$ watt/m^2
(Comment: The correct formula is $I = 10^{-12} \times 10^{0.1L} = 10^{0.1L-12}$)

Q10aii $I = 10^{-12} \times e^{0.1L} = 8.1 \times 10^{-9}$, $e^{0.1L} = 8.1 \times 10^3$

$L = \dfrac{\ln\left(8.1 \times 10^3\right)}{0.1} \approx 90$ decibels

Q10aiii $I = 10^{-12} \times e^{0.1L}$, $L = 10\ln\left(10^{12} I\right)$
Let $L_i = 10\ln\left(10^{12} I_i\right)$ and $L_f = 10\ln\left(10^{12} I_f\right)$

$\Delta L = L_f - L_i = 10\ln\left(10^{12} I_f\right) - 10\ln\left(10^{12} I_i\right) = 10\ln\dfrac{I_f}{I_i}$

If $I_f = 2I_i$, $\Delta L \approx 6.9$ decibels

Q10bi *Length of fencing P = arc length + 2 times radius*
$\therefore P = r\theta + 2r = r(\theta + 2)$

Q10bii $r\theta = P - 2r$
Sector area $A = \dfrac{1}{2} r^2 \theta = \dfrac{1}{2} r(P - 2r) = \dfrac{1}{2} P.r - r^2$

Q10biii Let $\dfrac{dA}{dr} = \dfrac{1}{2}P - 2r = 0$, $r = \dfrac{1}{4}P$ to maximise A.

Q10biv $P = r(\theta + 2)$, $P = \dfrac{1}{4}P(\theta + 2)$, $1 = \dfrac{1}{4}(\theta + 2)$, $\theta = 2$

Q10bv Since $0 < \theta < 2\pi$, $0 < r\theta < 2\pi r$,
$0 + 2r < r\theta + 2r < 2\pi r + 2r$, $\therefore 2r < P < 2r(\pi + 1)$

$\therefore r < \dfrac{P}{2}$ and $\dfrac{P}{2(\pi + 1)} < r$

$\therefore \dfrac{P}{2(\pi + 1)} < r < \dfrac{P}{2}$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.