2011 VCAA Math. Methods (CAS) Exam 1 Solutions

Q1a $\frac{d}{dx}\sqrt{4-x} = -\frac{1}{2\sqrt{4-x}}$

Q1b $g(x) = x^2 \sin 2x$, $g'(x) = 2x\sin 2x + 2x^2 \cos 2x$ (the product rule).

$g'\left(\frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right)\sin 2\left(\frac{\pi}{6}\right) + 2\left(\frac{\pi}{6}\right)^2 \cos 2\left(\frac{\pi}{6}\right) = \frac{\pi}{6}\left(\sqrt{3} + \frac{\pi}{6}\right)$

Q2a $\int \frac{1}{3x-4}\,dx = \frac{1}{3}\log_e|3x-4|$

Q2b $4^x - 15\times 2^x - 16 = 0$, $\left(2^x\right)^2 - 15\left(2^x\right) - 16 = 0$

$\left(2^x + 1\right)\left(2^x - 16\right) = 0$

Since $2^x + 1 > 0$, .: $2^x - 16 = 0$, $x = 4$

Q3a Range is $[1,7]$, period $= \frac{2\pi}{\frac{\pi}{2}} = 4$

Q3b $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ and $0 \le x \le \pi$, i.e. $\frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{7\pi}{3}$,

.: $2x + \frac{\pi}{3} = \frac{5\pi}{6}$ or $\frac{13\pi}{6}$, $2x = \frac{\pi}{2}$ or $\frac{11\pi}{6}$, .: $x = \frac{\pi}{4}$ or $\frac{11\pi}{12}$

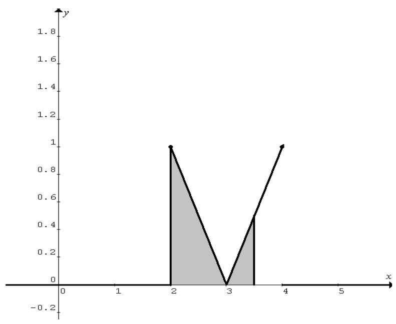
Q4a $f(x) = \sqrt{x^2 - 9}$, $g(x) = x + 5$,

$f(g(x)) = \sqrt{(x+5)^2 - 9} = \sqrt{(x+2)(x+8)}$

.: $c = 2$ and $d = 8$ OR $c = 8$ and $d = 2$

Q4b $(x+2)(x+8) \ge 0$, .: $x \le -8$ OR $x \ge -2$

Q5a



$\Pr(X < 3.5) = 0.5 + \frac{1}{4}\times 0.5 = 0.625$

Q5b $\Pr(X < 2.5 \mid X < 3.5) = \frac{\Pr(X < 2.5)}{\Pr(X < 3.5)} = \frac{\Pr(X > 3.5)}{\Pr(X < 3.5)} = \frac{0.375}{0.625} = 0.6$

Q6a $kx - 3y = k + 3$, .: $y = \frac{k}{3}x + \frac{-k-3}{3}$

$4x + (k+7)y = 1$, .: $y = \frac{-4}{k+7}x + \frac{1}{k+7}$

Infinitely many solutions: $\frac{k}{3} = \frac{-4}{k+7}$ and $\frac{-k-3}{3} = \frac{1}{k+7}$

.: $k^2 + 7k + 12 = 0$ and $k^2 + 10k + 24 = 0$

.: $(k+3)(k+4) = 0$ and $(k+6)(k+4) = 0$

.: only $k = -4$ satisfies both constraints.

Q6b Unique solution: $\frac{k}{3} \ne \frac{-4}{k+7}$, .: $(k+3)(k+4) \ne 0$

.: $k \ne -3$, -4, i.e. $k \in R \setminus \{-4,-3\}$

Q7ai $\Pr(X = 3) = p^3$

Q7aii $\Pr(X = 2) = 3p^2(1-p)$

Q7b $p^3 = 3p^2(1-p)$ where $0 < p < 1$

Since $p \ne 0$, $p = 3(1-p)$, $p = 3 - 3p$, $4p = 3$, $p = \frac{3}{4}$

Q8a $\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$,

also $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

.: $\Pr(A' \cap B) = \Pr(A \cup B) - \Pr(A) = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$

Q8b If A and B are mutually exclusive, $\Pr(A \cap B) = 0$,

.: $\Pr(A' \cap B) = \Pr(B) = \frac{1}{4}$

Q9 $f(x) = x^3 - ax$ and $g(x) = ax$

.: $f(x) + g(x) = x^3$

At $x = m \ne 0$, $f(m) = g(m)$ and $f(m) + g(m) = m^3$

.: $m^3 = 2g(m) = 2am$, .: $m^2 = 2a$ …… (1)

Area $= \int_0^m \left(ax - \left(x^3 - ax\right)\right)dx = 64$, .: $\left[ax^2 - \frac{x^4}{4}\right]_0^m = 64$,

$am^2 - \frac{m^4}{4} = 64$ …… (2) Solve (1) and (2), $a = 8$ and $m = 4$

Q10a $BD = a\cos\theta$, $CD = a\sin\theta$

Q10b $L = 4 + a + a\sin\theta + 2a\cos\theta$

Q10c $\frac{dL}{d\theta} = a\cos\theta - 2a\sin\theta$

Given $BD = 2CD$, i.e. $a\cos\theta - 2a\sin\theta = 0$, .: $\frac{dL}{d\theta} = 0$

Q10d Let $CD = x > 0$, .: $x^2 + (2x)^2 = \left(3\sqrt{5}\right)^2$,

.: $x^2 + 4x^2 = 45$, .: $x = 3$. L is maximum when $BD = 2CD$.

.: max $L = 2 + 3\sqrt{5} + 3 + 2 + 6 + 6 = 19 + 3\sqrt{5}$

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