2011 VCAA Math. Methods Exam 2 Solutions

Note: Some steps can be done by CAS

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
A	A	B	A	D	E	C	D	B	C	E

12	13	14	15	16	17	18	19	20	21	22
D	A	D	B	D	B	D	D	E	E	B

Q1 Midpoint: $\left(\frac{0+d}{2},\frac{-5+0}{2}\right)=\left(\frac{d}{2},-\frac{5}{2}\right)$ A

Q2 Gradient of given line: $\frac{0-(-4)}{(-2)-0}=-2$

.: gradient of perpendicular line is $\frac{1}{2}$. A

Q3 $f(x)=4x^3-13x^2-ax$ where $a\neq 0$

$f(-a)=-4a^3-13a^2+a^2=0$, .: $-4a^3-12a^2=0$

$-4a^2(a+3)=0$, .: $a=-3$ B

Q4 $\frac{d}{dx}\log_e(2f(x))=\frac{1}{2f(x)}\times 2f'(x)=\frac{f'(x)}{f(x)}$ A

Q5 The range of $y=\sqrt{2x-4}$ is $[0,\infty)$ and it is the domain of the inverse. Equation of the inverse: $x=\sqrt{2y-4}$, .: $y=\frac{x^2+4}{2}$

.: the rule is $g^{-1}(x)=\frac{x^2+4}{2}$ D

Q6 $E(X)=\int_1^e x\log_e(x)dx\approx 2.097$ E

Q7 $f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$, $f'(x)=\frac{1}{3x^{\frac{2}{3}}}=\frac{1}{3\times\left(\sqrt[3]{x}\right)^2}$

$f(8+h)\approx f(8)+hf'(8)$,

$\sqrt[3]{8.5}\approx\sqrt[3]{8}+0.5\times\frac{1}{3\times\left(\sqrt[3]{8}\right)^2}\approx 2.04167$ C

Q8 $h(x)=f(x)+g(x)=x(x-4)+x+3=x^2-3x+3$ and $x\in\left[\frac{3}{2},5\right)$. The range of h is $\left[\frac{3}{4},13\right)$, .: the domain of h^{-1} is $\left[\frac{3}{4},13\right)$. D

Q9 The stationary points of $f(x)$ are at $x=-3$ and 0, .: the graph of $f'(x)$ has x-intercepts at $x=-3$ and 0. $f(x)$ looks like a negative quartic, .: $f'(x)$ should be like a negative cubic. B

Q10 $W\to W$ $(pr=0.8)$; $L\to L$ $(pr=0.6)$

The transition matrix is $\begin{matrix} & W & L \\ W & 0.8 & 0.4 \\ L & 0.2 & 0.6\end{matrix}$, the long-term state matrix is $\begin{matrix} W \\ L\end{matrix}\begin{bmatrix}\frac{0.4}{0.4+0.2}\\ \frac{0.2}{0.4+0.2}\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}$ C

Q11 Average value $=\frac{\int_0^3\log_e(x+2)dx}{3-0}=\frac{5\log_e(5)-2\log_e(2)-3}{3}$ E

Q12 Solve for a by CAS. D

Q13 $\Pr(X>2.8)=\frac{1735}{2000}$, .: $\Pr(X<2.8)=\frac{265}{2000}=0.1325$

$\Pr\left(Z<\frac{2.8-\mu}{0.2}\right)=0.1325$

Standard inverse normal: $\frac{2.8-\mu}{0.2}\approx -1.11465$, $\mu\approx 3.023$ A

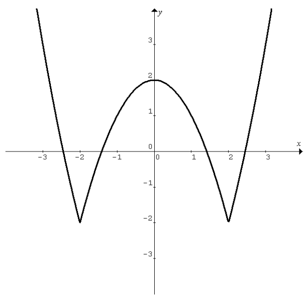
Q14 ii is correct, the shaded area equals the rectangular area minus the area under the curve.

iv is also correct: the inverse of $y=e^{2x}$ is $y=\frac{\log_e x}{2}$, the shaded area is under the curve $y=\frac{\log_e x}{2}$. D

Q15 Amplitude $=2$, period $=\frac{2\pi}{3}-\frac{\pi}{6}=\frac{\pi}{2}=\frac{2\pi}{n}$, .: $n=4$.

Translated upwards by 1 unit and to the right by $\frac{\pi}{6}$. B

Q16 The graph of $f(x)=\left|x^2-4\right|-2$:

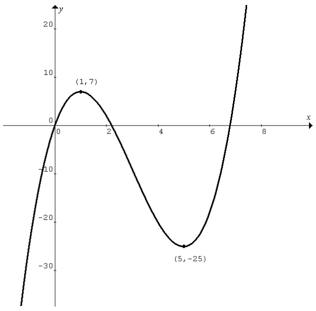


$f'(x)$ is negative for $0<x<2$ and positive for $x>2$. D

Q17 At $(4,12)$ gradient of the tangent: $m_t = \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 1 = 4$

.: gradient of the normal $m_n = -\frac{1}{4}$, same gradient as the line with equation $4y + x = 7$. B

Q18 Consider the graph of $f(x) = x^3 - 9x^2 + 15x$.



The local max and min are $(1,7)$ and $(5,-25)$ respectively. To Pass through the *x*-axis once, the graph must be lowered by more than 7 units or lifted up by more than 25 units.
.: $w < -7$ or $w > 25$. D

Q19 $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, $f(5) = 25$, $f(6) = 36$
Estimated area $= (1 + 4 + 9 + 16 + 25 + 36) \times 1 = 91$

Exact area $= \int_0^6 x^2 dx = \left[\frac{x^3}{3}\right]_0^6 = 72$

$p\% = \frac{91 - 72}{72} \times 100\% \approx 26.4\%$ D

Q20 $\int_0^a (x^2 - 4) dx = 0$ where $a > 0$,

.: $\left[\frac{x^3}{3} - 4x\right]_0^a = 0$, $\frac{a^3}{3} - 4a = 0$, $a^2 = 12$, $a = 2\sqrt{3}$ E

Q21 Given $\Pr(P \cap Q) = \Pr(P' \cap Q)$, and by the law of total probability, $\Pr(P \cap Q) + \Pr(P' \cap Q) = \Pr(Q)$.

.: $2 \times \Pr(P \cap Q) = \Pr(Q)$, $\Pr(P \cap Q) = \frac{1}{2}\Pr(Q)$.

For *P* and *Q* to be independent, $\Pr(P \cap Q) = \Pr(P) \times \Pr(Q)$,

.: $\frac{1}{2}\Pr(Q) = \Pr(P) \times \Pr(Q)$, .: $\Pr(P) = \frac{1}{2}$ E

Q22 $\log_c(a) + \log_a(b) + \log_b(c) = \frac{\log_a(a)}{\log_a(c)} + \frac{\log_b(b)}{\log_b(a)} + \frac{\log_c(c)}{\log_c(b)}$

$= \frac{1}{\log_a(c)} + \frac{1}{\log_b(a)} + \frac{1}{\log_c(b)}$ B

SECTION 2

Q1a Given $V = 6075$ L at $t = 0$ min, and $\frac{dV}{dt} = \frac{t^2}{5}$.
Let *T* be the required time.

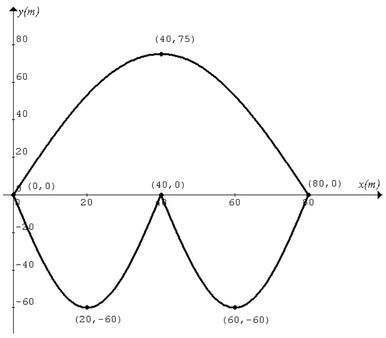
$V = \int_0^T \frac{t^2}{5} dt = 6075$, $\left[\frac{t^3}{15}\right]_0^T = 6075$, $T = 45$ min

Q1b $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, .: $\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$

Given $\frac{dA}{dt} = 20$ m^2 per min, when $r = 3$ m, $20 = 2\pi(3) \times \frac{dr}{dt}$,

.: $\frac{dr}{dt} = \frac{10}{3\pi}$ m per min.

Q1c



Q1di $\int_0^{40} \sin\left(\frac{\pi x}{40}\right) dx = \left[\frac{-40}{\pi}\cos\left(\frac{\pi x}{40}\right)\right]_0^{40} = \frac{80}{\pi}$ or 25.5 approx.

Q1dii Total area $= \int_0^{80} 75\sin\left(\frac{\pi x}{80}\right) dx + 2 \times 60 \times \frac{80}{\pi} \approx 6875$ m^2

Q2ai *X* has a normal distribution with $\mu = 3$ and $\sigma = 0.8$,
$\Pr(3 \le X \le 5) \approx 0.4938$

Q2aii $\Pr(3 \le Y \le 5) = \int_3^4 \frac{y}{16} dy + \int_4^5 0.25e^{-0.5(y-4)} dy \approx 0.4155$

Q2b Mean of $Y = \int_0^4 y \times \frac{y}{16} dy + \int_4^\infty y \times 0.25e^{-0.5(y-4)} dy \approx 4.333$

Q2ci Since $\int_0^4 \frac{y}{16} dy = 0.5$, .: the median of *Y* is 4.

Q2cii $\Pr(Y \le a) = 0.7$, .: $\Pr(4 \le Y \le a) = 0.2$

Solve $\int_4^a 0.25e^{-0.5(y-4)} dy = 0.2$ for a, $a \approx 5.02$

Q2d Binomial: $n = 10$, $p = \frac{9}{32}$, $\Pr(X = 4) \approx 0.1812$

Q2e Let X be the time required to produce a chocolate.

Given $\Pr(X \le 3 \mid A) = 0.5$ and $\Pr(X \le 3 \mid B) = \frac{9}{32}$

.: $\Pr(X > 3 \mid A) = 0.5$ and $\Pr(X > 3 \mid B) = 1 - \frac{9}{32} = \frac{23}{32}$.

Also given $\Pr(A) = \Pr(B) = 0.5$.

.: $\Pr(X > 3 \cap A) = \Pr(X > 3 \mid A) \times \Pr(A) = 0.5 \times 0.5 = 0.25$

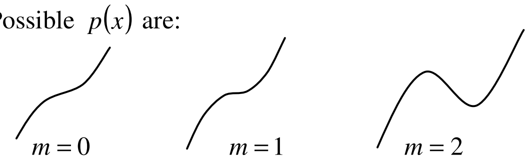
$\Pr(X > 3 \cap B) = \Pr(X > 3 \mid B) \times \Pr(B) = \frac{23}{32} \times 0.5 = \frac{23}{64}$

.: $\Pr(A \mid X > 3) = \frac{\Pr(X > 3 \cap A)}{\Pr(X > 3)} = \frac{0.25}{0.25 + \frac{23}{64}} \approx 0.4103$

Q3ai $f(x) = 4x^3 + 5x - 9$, $f'(x) = 12x^2 + 5$

Q3aii Since $x^2 \ge 0$ for all x, .: $12x^2 \ge 0$, .: $12x^2 + 5 \ge 0 + 5$,
.: $f'(x) \ge 5$ for all x.

Q3bi Possible $p(x)$ are:



Q3bii The first two cases ($m = 0$ and $m = 1$) above are one-to-one functions, .: each has an inverse function.

Q3ci Let $q(x) = 3 - 2x^3 = y$, the equation of the inverse is

$3 - 2y^3 = x$, .: $y^3 = \frac{3 - x}{2}$, $y = \left(\frac{3 - x}{2}\right)^{\frac{1}{3}}$, .: $q^{-1}(x) = \left(\frac{3 - x}{2}\right)^{\frac{1}{3}}$

Q3cii At the intersection $y = x$, .: $x = 3 - 2x^3$, .: $x = 1$ and $y = 1$. The point of intersection is $(1,1)$.

Q3di $g(x) = x^3 + 2x^2 + cx + k$, $g'(x) = 3x^2 + 4x + c$

For $g'(x) = 0$ at exactly one point, $\Delta = 4^2 - 4(3)c = 0$, .: $c = \frac{4}{3}$

Q3dii $g(x) = x^3 + 2x^2 + \frac{4}{3}x + k$. Let $g'(x) = 3x^2 + 4x + \frac{4}{3} = 0$,

.: the stationary is at $x = -\frac{2}{3}$.

At the intersection of $y = g(x)$ and $y = g^{-1}(x)$, $y = x$,

.: $x^3 + 2x^2 + \frac{4}{3}x + k = x$,

$\left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 + \frac{4}{3}\left(-\frac{2}{3}\right) + k = -\frac{2}{3}$, .: $k = -\frac{10}{27}$

Q4a $y = x^2 - 1$, $x \ge 0$, .: $n = m^2 - 1$

$L = \sqrt{(m-0)^2 + (n-1)^2} = \sqrt{(m-0)^2 + (m^2 - 2)^2}$

.: $L = \sqrt{m^4 - 3m^2 + 4}$ where $m > 0$

Q4bi $\frac{dL}{dm} = \frac{1}{2\sqrt{m^4 - 3m^2 + 4}} \times (4m^3 - 6m) = \frac{m(2m^2 - 3)}{\sqrt{m^4 - 3m^2 + 4}}$

Let $\frac{dL}{dm} = 0$, .: $2m^2 - 3 = 0$, $m = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

Coordinates of plant $= (m, n) = \left(\frac{\sqrt{6}}{2}, \frac{1}{2}\right)$

Q4bii $L = \sqrt{m^2 + (n-1)^2} = \sqrt{\frac{3}{2} + \frac{1}{4}} = \frac{\sqrt{7}}{2}$ km

Q4c Distance from the camp to point $(x, y) = \sqrt{x^2 + (x^2 - 1)^2}$,

.: time taken from the camp to $(x, y) = \frac{\sqrt{x^2 + (x^2 - 1)^2}}{2}$.

Time taken from (x, y) to the plant $= k\left(\frac{3}{4} - (x^2 - 1)\right)$.

Total time taken $T = \frac{\sqrt{x^2 + (x^2 - 1)^2}}{2} + k\left(\frac{3}{4} - (x^2 - 1)\right)$

.: $T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k(7 - 4x^2)$ hours

Q4di $\frac{dT}{dx} = \frac{2x^3 - x}{2\sqrt{x^4 - x^2 + 1}} - \frac{x}{\sqrt{13}}$ when $k = \frac{1}{2\sqrt{13}}$

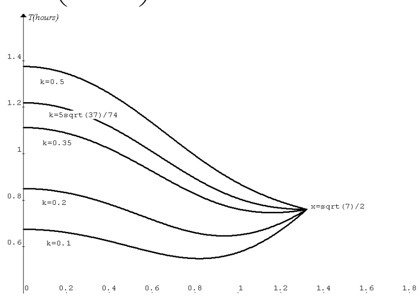
Q4dii Let $\frac{dT}{dx} = \frac{2x^3 - x}{2\sqrt{x^4 - x^2 + 1}} - \frac{x}{\sqrt{13}} = 0$, $x = \frac{\sqrt{3}}{2}$ and

$y = x^2 - 1 = -\frac{1}{4}$. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$

Q4e $\frac{dT}{dx} = \frac{2x^3 - x}{2\sqrt{x^4 - x^2 + 1}} - 2kx = 0$ at $(1,0)$, .: $k = \frac{1}{4}$

Q4f $\frac{dT}{dx} = \frac{2x^3 - x}{2\sqrt{x^4 - x^2 + 1}} - 2kx = 0$ at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$, .: $k = \frac{5\sqrt{37}}{74}$.

If $k > \frac{5\sqrt{37}}{74}$, $T(x)$ is a decreasing function in $\left[0, \frac{\sqrt{7}}{2}\right]$, .: $T(x)$ is a minimum at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$. .: $k \ge \frac{5\sqrt{37}}{74}$



Please inform mathline@itute.com re conceptual, mathematical and/or typing errors