2010 VCAA Specialist Math Exam 2 Solutions

Note: Some steps can be done by CAS

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	B	E	D	C	B	C	E	A	C	E

12	13	14	15	16	17	18	19	20	21	22
C	C	D	D	A	A	B	A	E	B	E

Q1 $y=\frac{2x^3+x^2-1}{x^2-x-2}=2x+3+\frac{7x+5}{(x-2)(x+1)}$

Straight line asymptotes are: $y=2x+3$, $x=2$ and $x=-1$ D

Q2 Circle $x^2-6x+y^2+4y=b$, centre $(a,-2)$, radius 5

$x^2-6x+9+y^2+4y+4=b+9+4$

$(x-3)^2+(y+2)^2=b+13$

.: $a=3$ and $b+13=25$, $b=12$ B

Q3 $-1\le ax\le 1$, $-\frac{1}{a}\le x\le\frac{1}{a}$ E

Q4 $\frac{(y+3)^2}{9}-\frac{(x-6)^2}{4}=1$, $\frac{(x-6)^2}{2^2}-\frac{(y+3)^2}{3^2}-=-1$

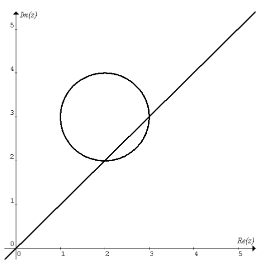
Asymptotes: $y+3=\pm\frac{3}{2}(x-6)$

.: $y=\frac{3}{2}x-12$ and $y=-\frac{3}{2}x+6$ D

Q5 $\frac{1}{\bar{z}}=\frac{1}{\bar{z}}\times\frac{z}{z}=\frac{z}{|z|^2}$, .: $\frac{1}{\bar{z}}$ has the same argument as z but a smaller modulus, given $|z|>1$. C

Q6 $P(z)=0$ has real coefficients, .: the conjugate roots theorem can be applied to it. Given $z=3i$ as one of the roots, .: $z=-3i$ is also a root. B

Q7 C



Q8 $z^3=-27i=(3i)^3$, .: one of the three roots of $z^3=-27i$ is $3i$. The three roots are evenly spaced around the circle of radius 3. The points which represent the roots are P_4, P_8 and P_{12}. E

Q9 $x\sin(x)\sec(2x)=\frac{x\sin(x)}{\cos(2x)}=0$, $x\in[0,\ 2\pi]$

.: $x=0$, $\sin(x)=0$ and $\cos(2x)\neq 0$

.: $x=0$, π or 2π A

Q10 The diagonals of a rhombus are perpendicular.

.: $(\tilde{a}+\tilde{b}).(\tilde{a}-\tilde{b})=0$ C

Q11 $\tilde{F}_1+\tilde{F}_2+\tilde{F}_3=\frac{1}{2}\tilde{i}+\frac{\sqrt{3}}{2}\tilde{j}$, .: $|\tilde{F}_1+\tilde{F}_2+\tilde{F}_3|=|\tilde{F}_3|=1$ E

Q12 $\cos\theta=\frac{(3\tilde{i}+6\tilde{j}-2\tilde{k}).(2\tilde{i}-2\tilde{j}+\tilde{k})}{|3\tilde{i}+6\tilde{j}-2\tilde{k}||2\tilde{i}-2\tilde{j}+\tilde{k}|}=-\frac{8}{21}$

.: $\theta=\cos^{-1}\left(-\frac{8}{21}\right)\approx 112.4°$ C

Q13 $\tilde{r}(t)=(\sqrt{t-2})\tilde{i}+(2t)\tilde{j}$, $t\ge 2$

$y=2t$ and $x=\sqrt{t-2}$, .: $x\ge 0$ and $t=x^2+2$

.: $y=2(x^2+2)=2x^2+4$, $x\ge 0$ C

Q14 $f(0)=0$, either B or D.

$f'(0)=0$ for both also.

Only D gives $f''(x)=2e^x\sin(x)$ D

Q15 Let $u=\tan(2x)$, .: $\frac{1}{2}\frac{du}{dx}=\sec^2(2x)$

When $x=0$, $u=0$; when $x=\frac{\pi}{24}$, $u=2-\sqrt{3}$

.: $\int_0^{\frac{\pi}{24}}\tan(2x)\sec^2(2x)dx=\frac{1}{2}\int_0^{2-\sqrt{3}}(u)du$ D

Q16 *Gradient of the tangent* $=\frac{dy}{dx}$,

gradient of the normal $=-\frac{1}{\frac{dy}{dx}}=2\times\frac{y-1}{x-1}$

.: $\frac{dy}{dx}=-\frac{x-1}{2(y-1)}$, .: $\frac{dy}{dx}+\frac{x-1}{2(y-1)}=0$ A

Q17 The gradient of the little line segment at $(1,0)$ is 2, i.e. $\frac{dy}{dx}=-2$. Only $\frac{dy}{dx}=\frac{y-2x}{2y+x}$ satisfies this condition. A

Q18 $\frac{dx}{dt}=-\frac{10}{10-t}$

$t=0$, $x=5$, $\frac{dx}{dt}=-1$

$t=0.5$, $x=5+0.5(-1)=4.5$, $\frac{dx}{dt}=-\frac{10}{10-0.5}=-1.05263$

$t=1.0$, $x=4.5+0.5(-1.05263)\approx 3.97$ B

Q19 *Position = area of the trapezium – the area of the triangle*

$=\frac{1}{2}(5+15)3-\frac{1}{2}\times 10\times 2.5=17.5\text{ m}$ A

Q20 Given $v=2\sqrt{1-x^2}$, $v^2=4(1-x^2)$,

$a=\frac{1}{2}\times\frac{d}{dx}(v^2)=-4x$ E

Q21 $u=0$, $v=12$, $s=16$, $a=?$

$v^2=u^2+2as$, $a=4.5$, $F=ma=2\times 4.5=9.0\text{ N}$ B

Q22 $v=\frac{dx}{dt}=25+x^2$, $\frac{dt}{dx}=\frac{1}{5^2+x^2}$, $t=\int_5^x\frac{1}{5^2+x^2}dx+0$,

.: $t=\frac{1}{5}\int_5^x\frac{5}{5^2+x^2}dx=\frac{1}{5}\left[\tan^{-1}\left(\frac{x}{5}\right)\right]_5^x$

$=\frac{1}{5}\left(\tan^{-1}\left(\frac{x}{5}\right)-\frac{\pi}{4}\right)=\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)-\frac{\pi}{20}$ E

SECTION 2

Q1a $|z-i|=1$, $|x+yi-i|=1$, $|x+yi-i|^2=1$, $|x+(y-1)i|^2=1$,

.: $x^2+(y-1)^2=1$

Q1b The 2 circles are reflections of each other in the line $y=x$,

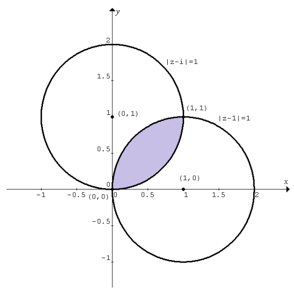
.: the intersections are on $y=x$.

Solve $x^2+(y-1)^2=1$ and $y=x$, .: $x^2+(x-1)^2=1$,

$2x^2-2x=0$, .: $x=0$ and $y=0$ OR $x=1$ and $y=1$

The points of intersection are $(0,0)$ and $(1,1)$.

Q1c and ei



Q1di $y=x$

Q1dii $z=a\bar{z}$, $x+yi=a(x-yi)$,

.: $1+i=a(1-i)$ since $(1,1)$ is on it.

.: $a=\frac{1+i}{1-i}=i$

Q1eii *Area of the shaded region* $=2\left(\frac{1}{4}\times\pi\times 1^2-\frac{1}{2}\right)=\frac{\pi}{2}-1$

Q2a Given $\tilde{a}(t)=-9.8\tilde{k}$, $\tilde{v}(0)=35\tilde{i}+5\tilde{j}+24.5\tilde{k}$, $\tilde{r}(0)=\tilde{0}$.

$\tilde{v}(t)=-9.8t\tilde{k}+\tilde{c}$, .: $\tilde{c}=35\tilde{i}+5\tilde{j}+24.5\tilde{k}$

.: $\tilde{v}(t)=-9.8t\tilde{k}+35\tilde{i}+5\tilde{j}+24.5\tilde{k}=35\tilde{i}+5\tilde{j}+(24.5-9.8t)\tilde{k}$

.: $\tilde{r}(t)=35t\tilde{i}+5t\tilde{j}+(24.5t-4.9t^2)\tilde{k}+\tilde{d}$, $\tilde{d}=0$ since $\tilde{r}(0)=\tilde{0}$.

.: $\tilde{r}(t)=35t\tilde{i}+5t\tilde{j}+(24.5t-4.9t^2)\tilde{k}$

Q2b The k-component of $\tilde{r}(t)$ is zero at the start and finish.

.: let $24.5t-4.9t^2=0$, $t=0$ or 5. .: $\Delta t=5-0=5$ seconds

Q2c At $t=\frac{5}{2}=2.5$,

maximum height $=24.5\times 2.5-4.9(2.5)^2=30.625\text{ m}$

Q2d $\tilde{v}(t)=35\tilde{i}+5\tilde{j}+(24.5-9.8t)\tilde{k}$,

.: $\tilde{v}(5)=35\tilde{i}+5\tilde{j}-24.5\tilde{k}$ is the velocity when the ball hits the ground. *Speed* $=|\tilde{v}(5)|=\sqrt{35^2+5^2+(-24.5)^2}\approx 43\text{ ms}^{-1}$

Q2e $\tilde{r}_{hole}=200\tilde{i}$, $\tilde{r}_{ball}(5)=175\tilde{i}+25\tilde{j}$

Distance from the hole to the ball $=|\tilde{r}_{ball}(5)-\tilde{r}_{hole}|$

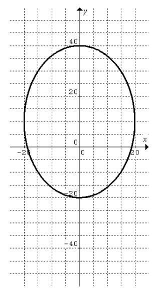
$=|-25\tilde{i}+25\tilde{j}|=\sqrt{(-25)^2+25^2}\approx 35\text{ m}$

Q3a x-intercepts:

Let $y=0$, $\frac{x^2}{400}+\frac{1}{9}=1$, $x^2=\frac{3200}{9}$,

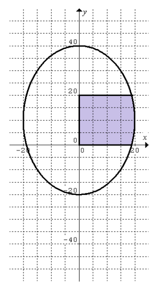
$x=\pm\frac{40\sqrt{2}}{3}$.

x-ntercepts $\left(-\frac{40\sqrt{2}}{3},0\right)$, $\left(\frac{40\sqrt{2}}{3},0\right)$



Q3bi

$V=\int_0^{20}\pi x^2dy=\int_0^{20}\pi\left(400-\frac{4(y-10)^2}{9}\right)dy$



Q3bii $V\approx 24202\text{ cm}^3$

Q3c Given $\frac{dV}{dt}=500$ and $\frac{dV}{dh}=\frac{25\pi}{36}(800+20h-h^2)$.

$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, .: $500=\frac{25\pi}{36}(800+20h-h^2)\times\frac{dh}{dt}$.

When $h=15$, $\frac{dh}{dt}=\frac{500}{\frac{25\pi}{36}(800+20(15)-15^2)}\approx 0.26\text{ cm min}^{-1}$

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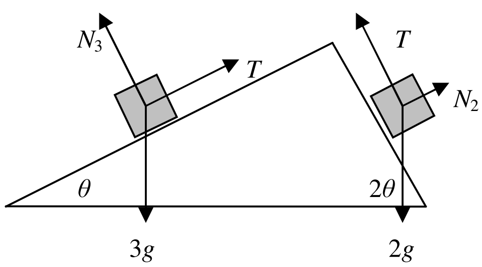
Q4



Q4a $T - 3g\sin\theta = 3a$ ………………… (1)

Q4b $2g\sin(2\theta) - T = 2a$ ……………… (2)

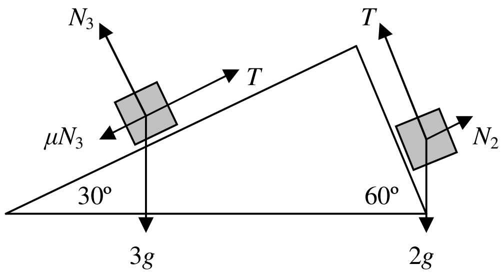
Q4c (1) + (2): $2g\sin(2\theta) - 3g\sin\theta = 5a$

$4g\sin\theta\cos\theta - 3g\sin\theta = 5a$, .: $a = \frac{g\sin\theta}{5}(4\cos\theta - 3)$

Q4d Let $a = 0$ for equilibrium. .: $\frac{g\sin\theta}{5}(4\cos\theta - 3) = 0$

Since $0 < \theta < \frac{\pi}{2}$, .: $4\cos\theta - 3 = 0$, $\theta = \cos^{-1}\left(\frac{3}{4}\right) \approx 41.4°$

Q4e



$N_3 = 3g\cos 30° = \frac{3\sqrt{3}g}{2}$, $\mu N_3 = 0.05 \times \frac{3\sqrt{3}g}{2} = \frac{3\sqrt{3}g}{40}$

3-kg mass: $T - 3g\sin 30° - \mu N_3 = 3a$,

.: $T - \frac{3g}{2} - \frac{3\sqrt{3}g}{40} = 3a$ …………………… (1)

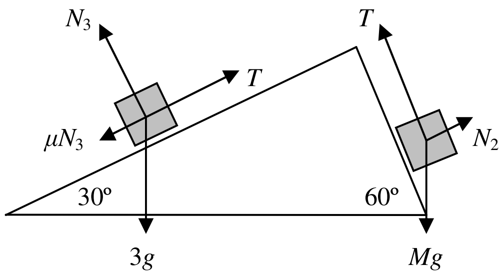
2-kg mass: $2g\sin 60° - T = 2a$

.: $\sqrt{3}g - T = 2a$ …………………………… (2)

Solve (1) and (2): $a \approx 0.2$ m s^{-2}

Q4f

$M = m_2$

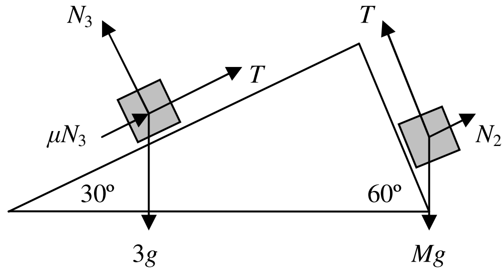


3-kg mass: $T - 3g\sin 30° - \mu N_3 = 0$, $T - 15.97306 = 0$ … (1)

M mass: $Mg\sin 60° - T = 0$, $8.487M - T = 0$ …………..... (2)

Solve (1) and (2): $M \approx 1.88$, i.e. $m_2 \approx 1.88$ kg

$M = m_1$



3-kg mass: $3g\sin 30° - \mu N_3 - T = 0$, $13.427 - T = 0$ …… (1)

M mass: $T - Mg\sin 60° = 0$, $T - 8.487M = 0$ ……….…. (2)

Solve (1) and (2): $M \approx 1.58$, i.e. $m_1 \approx 1.58$ kg

Q5a At time t min, $volume = 10t + 10$ litres, $mass = x$ grams,

.: $concentration = \frac{x}{10(t+1)}$ grams per litre

Q5b *Rate of inflow of chemical* $= 20 \times e^{-0.2t}$ grams per min,

rate of outflow of chemical $= \frac{x}{10(t+1)} \times 10 = \frac{x}{t+1}$ grams per min

Rate of change of the amount of chemical

= rate of inflow – rate of outflow

.: $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{t+1}$,

.: $\frac{dx}{dt} + \frac{x}{t+1} = 20e^{-0.2t}$.

Q5ci $x(t) = \frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}$

$$\frac{dx}{dt} = \frac{20e^{-0.2t}(t+6)}{t+1} + \frac{500e^{-0.2t} - 600}{(t+1)^2}$$

Q5cii Differential equation: $\frac{dx}{dt} + \frac{x}{t+1} = 20e^{-0.2t}$

$$LHS = \frac{20e^{-0.2t}(t+6)}{t+1} + \frac{500e^{-0.2t} - 600}{(t+1)^2} + \frac{\frac{600}{t+1} - \frac{100e^{-0.2t}(t+6)}{t+1}}{t+1}$$

$$= \frac{20e^{-0.2t}(t+6)(t+1)}{(t+1)^2} + \frac{500e^{-0.2t} - 600}{(t+1)^2} + \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2}$$

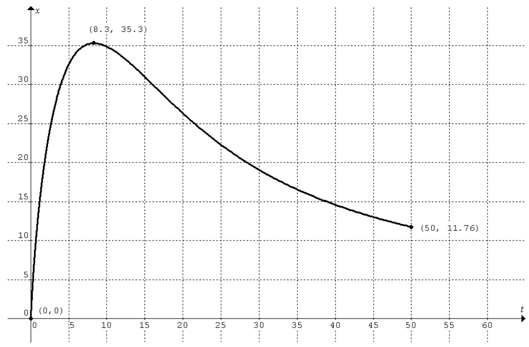
$$= \frac{20e^{-0.2t}(t+6)(t+1)}{(t+1)^2} + \frac{500e^{-0.2t}}{(t+1)^2} - \frac{100e^{-0.2t}(t+6)}{(t+1)^2}$$

$$= \frac{20e^{-0.2t}}{(t+1)^2}\left((t+6)(t+1) + 25 - 5(t+6)\right)$$

$$= \frac{20e^{-0.2t}}{(t+1)^2}(t+1)^2 = 20e^{-0.2t} = RHS$$

Initial condition: $x(0) = \frac{600}{0+1} - \frac{100e^{0}(0+6)}{0+1} = 0$

Q5d



Turning point: $(8.3, 35.3)$

Q5e $\textit{Rate of outflow} = \frac{x}{t+1} = \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2}$

$$\textit{Amount of outflow} = \int_0^{10} \frac{600 - 100e^{-0.2t}(t+6)}{(t+1)^2}\,dt \approx 51.6 \text{ grams}$$

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