2012 NSW BOS Mathematics Extension 2 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
D	D	A	A	C	B	A	B	C	B

Q1 $2z+\overline{w}=2(5-i)+2-3i=12-5i$ D

Q2 $x^3-y^3+3xy+1=0$, $\frac{d}{dx}\left(x^3-y^3+3xy+1\right)=0$

$3x^2-3y^2\frac{dy}{dx}+3y+3x\frac{dy}{dx}=0$

At $(1,2)$, $3-12\frac{dy}{dx}+6+3\frac{dy}{dx}=0$, $\frac{dy}{dx}=1$ D

Q3 $\overline{z}$ is the reflection of z in the x-axis, and $i\,\overline{z}$ is the anticlockwise rotation of $\overline{z}$ by $90°$. A

Q4 $y=[f(x)]^2$ has the same x-intercepts as $y=f(x)$.
At the x-intercepts, $y'=2f(x).f'(x)=0$. A

Q5 $2x^3-3x^2-5x-1=2(x-\alpha)(x-\beta)(x-\gamma)=0$

.: $\alpha\beta\gamma=\frac{1}{2}$, $\frac{1}{\alpha\beta\gamma}=2$, $\frac{1}{\alpha^3\beta^3\gamma^3}=8$ C

Q6 $b^2=a^2\left(e^2-1\right)$, $4=6\left(e^2-1\right)$, $e^2=\frac{5}{3}$, $e=\frac{\sqrt{5}}{\sqrt{3}}=\frac{\sqrt{15}}{3}$ B

Q7 Vertical component: $\sum F=0$, $T\cos\alpha+N-mg=0$
Horizontal component: $\sum F=ma$, $T\sin\alpha=mr\omega^2$ A

Q8 Let $P'(x)=\left(x+\frac{5}{4}\right)(x-1)^2$, .: $P(x)=(x-b)(x-1)^3$
.: either A or B. Only B yields the correct $P'(x)$. B

Q9 $\delta V=2\pi(x+2)y\delta x$

.: $V=\int_0^2 2\pi(x+2)y\,dx=2\pi\int_0^2(x+2)x(2-x)\,dx$ C

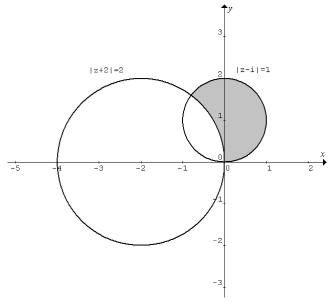
Q10 $x^3\sin x$ is an even function and it is ≥ 0 in $[-\pi,\pi]$

.: $\int_{-\pi}^{\pi}x^3\sin x\,dx>0$ B

Section II

Q11a $\frac{2\sqrt{5}+i}{\sqrt{5}-i}=\frac{\left(2\sqrt{5}+i\right)\left(\sqrt{5}+i\right)}{\left(\sqrt{5}-i\right)\left(\sqrt{5}+i\right)}=\frac{9+3\sqrt{5}i}{6}=\frac{3}{2}+\frac{\sqrt{5}}{2}i$

Q11b



Q11c $\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{1+(x+2)^2}=\tan^{-1}(x+2)+c$

Q11di $z=\sqrt{3}-i$, $|z|=\sqrt{3+1}=2$, $Arg(z)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

$z=2cis\left(-\frac{\pi}{6}\right)$

Q11dii $z^9=\left[2cis\left(-\frac{\pi}{6}\right)\right]^9=2^9cis\left(-\frac{3\pi}{2}\right)=2^9cis\left(\frac{\pi}{2}\right)=2^9i$

Q11e $\int_0^1\frac{e^{2x}}{e^{2x}+1}dx$

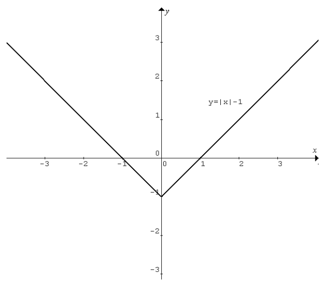
$=\frac{1}{2}\int_2^{e^2+1}\frac{du}{u}=\frac{1}{2}\left[\log_e|u|\right]_2^{e^2+1}$

$=\frac{1}{2}\left(\log_e\left(e^2+1\right)-\log_e 2\right)$

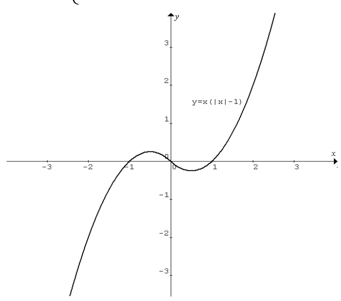
$=\frac{1}{2}\log_e\frac{e^2+1}{2}$

Let $u=e^{2x}+1$
$\frac{1}{2}\frac{du}{dx}=e^{2x}$
When $x=0$, $u=2$
When $x=1$,
$u=e^2+1$

Q11fi



Q11fii $y=x\left(|x|-1\right)=\begin{cases}-x^2-x, & x<0\\ x^2-x, & x\geq 0\end{cases}$



Q12a $\frac{1}{1-\cos\theta}=\frac{1}{2\sin^2\frac{\theta}{2}}=\frac{\sec^2\frac{\theta}{2}}{2\tan^2\frac{\theta}{2}}$

Let $t=\tan\frac{\theta}{2}$, $\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}$

$$\int\frac{d\theta}{1-\cos\theta}=\int\frac{\sec^2\frac{\theta}{2}d\theta}{2\tan^2\frac{\theta}{2}}=\int\frac{\frac{dt}{d\theta}d\theta}{t^2}=\int\frac{dt}{t^2}=-\frac{1}{t}=-\cot\frac{\theta}{2}+c$$

Q12bi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$

$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$, $y'=-\frac{b^2x}{a^2y}$, $m_{PT}=-\frac{b^2x_0}{a^2y_0}$

Tangent at P: $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$

Q12bii Normal at P: $y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$

At point N, $y=0$, $-1=\frac{a^2}{b^2x_0}(x-x_0)$, $x=x_0\left(1-\frac{b^2}{a^2}\right)=x_0e^2$

Q12biii (x_0,y_0) is on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, .: $\frac{x_0{}^2}{a^2}+\frac{y_0{}^2}{b^2}=1$,

.: $b^2x_0{}^2+a^2y_0{}^2=a^2b^2$

At point T, $y=0$, $-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$

.: $x=\frac{b^2x_0{}^2+a^2y_0{}^2}{b^2x_0}=\frac{a^2b^2}{b^2x_0}=\frac{a^2}{x_0}$

.: $ON\times OT=x_0e^2\times\frac{a^2}{x_0}=(ae)^2=OS^2$

Q12c $\frac{d}{dx}\left[x(\log_e x)^n\right]=n(\log_e x)^{n-1}+(\log_e x)^n$

$$\int_1^{e^2}\frac{d}{dx}\left[x(\log_e x)^n\right]dx=n\int_1^{e^2}(\log_e x)^{n-1}dx+\int_1^{e^2}(\log_e x)^n dx$$

$$\left[x(\log_e x)^n\right]_1^{e^2}=n\int_1^{e^2}(\log_e x)^{n-1}dx+\int_1^{e^2}(\log_e x)^n dx$$

$$e^2\left(\log_e e^2\right)^n=n\int_1^{e^2}(\log_e x)^{n-1}dx+\int_1^{e^2}(\log_e x)^n dx$$

.: $e^2 2^n=nI_{n-1}+I_n$, .: $I_n=e^2 2^n-nI_{n-1}$

Q12di w_1 is a 90° anti-clockwise rotation of z about u_1,
.: $w_1-u_1=i(z-u_1)$. Hence $w_1=u_1+i(z-u_1)$.

Q12dii w_2 is a 90° clockwise rotation of z about u_2,
.: $w_2-u_2=-i(z-u_2)$. Hence $w_2=u_2-i(z-u_2)$.

Midpoint of $B_1B_2=\frac{1}{2}(w_1+w_2)=\frac{1}{2}\left[u_1+u_2-i(u_1-u_2)\right]$ is independent of z, .: the locus is a fixed point.

Q13ai $\frac{dv}{dt}=10-\frac{v^2}{40}=\frac{400-v^2}{40}$, .: $\frac{dt}{dv}=\frac{40}{400-v^2}$

$$t=40\int\frac{1}{400-v^2}dv=40\int\frac{1}{(20-v)(20+v)}dv$$

$$=\int\left(\frac{1}{20-v}+\frac{1}{20+v}\right)dv=-\log_e(20-v)+\log_e(20+v)+c$$

.: $t=\log_e\frac{20+v}{20-v}+c$. At $t=0$, $v=0$, .: $t=\log_e\frac{20+v}{20-v}$,

$e^t=\frac{20+v}{20-v}$, .: $v=\frac{20(e^t-1)}{e^t+1}$

Q13aii $v\frac{dv}{dx}=\frac{400-v^2}{40}$, $\frac{dv}{dx}=\frac{400-v^2}{40v}$, $\frac{dx}{dv}=\frac{40v}{400-v^2}$,

$$x=\int\frac{40v}{400-v^2}dv$$

$$=-20\int\frac{\frac{du}{dv}dv}{u}=-20\int\frac{du}{u}$$

.: $-\frac{x}{20}=\log_e\left|400-v^2\right|+c$

Let $u=400-v^2$
$\frac{du}{dv}=-2v$,
$v=-\frac{1}{2}\frac{du}{dv}$

When $t=0$, $x=0$ and $v=0$

.: $c=-\log_e 400$ and $400-v^2>0$

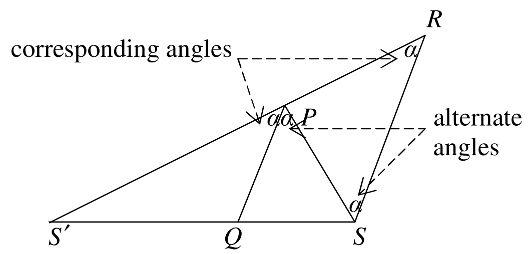
.: $-\frac{x}{20}=\log_e\frac{400-v^2}{400}$, $x=20\log_e\left(\frac{400}{400-v^2}\right)$

Q13aiii When $t=4$, $v=\frac{20(e^4-1)}{e^4+1}$, $\frac{v^2}{400}=\left(\frac{e^4-1}{e^4+1}\right)^2$

$$.:\ x=20\log_e\left(\frac{400}{400-v^2}\right)=20\log_e\left(\frac{1}{1-\left(\frac{e^4-1}{e^4+1}\right)^2}\right)$$

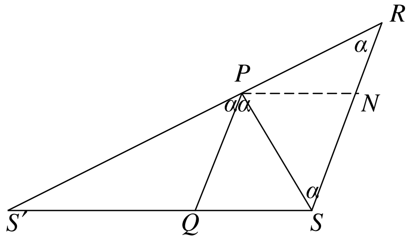
$$=20\log_e\frac{(e^4+1)^2}{(e^4+1)^2-(e^4-1)^2}=40\log_e\left(\frac{e^4+1}{2e^4}\right)$$

Q13bi



$\angle PRS=\angle PSR$, .: $PS=PR$

Q13bii



Construct PN parallel to QS.
.: $PNSQ$ is a parallelogram and $PN = QS$.

$\Delta PQS'$ and ΔRNP are similar, .: $\dfrac{PR}{PN} = \dfrac{PS'}{QS'}$, .: $\dfrac{PS}{QS} = \dfrac{PS'}{QS'}$

Q13ci $SP = ePM$, where M is on the directrix $x = \dfrac{a}{e}$ and PM is perpendicular to the directrix.

.: $SP = e\left(a\sec\theta - \dfrac{a}{e}\right) = a(e\sec\theta - 1)$

Q13cii Let $Q(q,0)$, $\dfrac{PS}{QS} = \dfrac{PS'}{QS'}$, .: $\dfrac{a(e\sec\theta - 1)}{ae - q} = \dfrac{a(e\sec\theta + 1)}{q + ae}$

$(q + ae)(e\sec\theta - 1) = (ae - q)(e\sec\theta + 1)$, .: $q = \dfrac{a}{\sec\theta}$

Q13ciii Gradient of $PQ = \dfrac{b\tan\theta}{a\sec\theta - \dfrac{a}{\sec\theta}} = \dfrac{b\tan\theta}{a\sec\theta - \dfrac{a}{\sec\theta}}$

$= \dfrac{b\tan\theta\sec\theta}{a(\sec^2\theta - 1)} = \dfrac{b\tan\theta\sec\theta}{a\tan^2\theta} = \dfrac{b\sec\theta}{a\tan\theta}$

.: line PQ is a tangent to the hyperbola at point P.

Q14a $\dfrac{3x^2 + 8}{x(x^2 + 4)} = \dfrac{A}{x} + \dfrac{Bx}{x^2 + 4}$, $A(x^2 + 4) + Bx^2 = 3x^2 + 8$

.: $A = 2$ and $B = 1$

$\int \dfrac{3x^2 + 8}{x(x^2 + 4)}dx = \int\left(\dfrac{2}{x} + \dfrac{x}{x^2 + 4}\right)dx$,

$= \int \dfrac{2}{x}dx + \dfrac{1}{2}\int\dfrac{du}{u}$

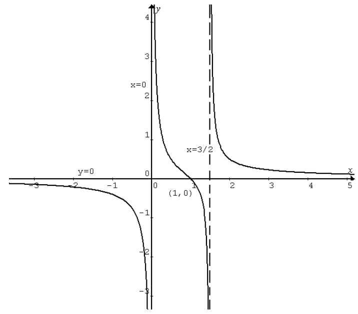
Let $u = x^2 + 4$, $\dfrac{1}{2}\dfrac{du}{dx} = x$

$= 2\log_e|x| + \dfrac{1}{2}\log_e(x^2 + 1) + c = \log_e\dfrac{x^2}{\sqrt{x^2 + 1}} + c$

Q14bi The x-intercepts of $y = \dfrac{x(2x - 3)}{x - 1}$ are $(0,0)$ and $\left(\dfrac{3}{2},0\right)$, and its vertical asymptote is $x = 1$.

$y = \dfrac{x - 1}{x(2x - 3)}$ is the reciprocal of $y = \dfrac{x(2x - 3)}{x - 1}$,

.: the x-intercept of $y = \dfrac{x - 1}{x(2x - 3)}$ is $(1,0)$, its vertical asymptotes are $x = 0$ and $x = \dfrac{3}{2}$, and it has a horizontal asymptote $y = 0$. The y-intercept does not exist.



Q14bii

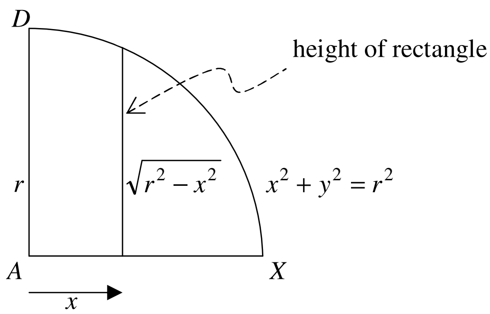
$$\begin{array}{r} 2x - 1 \\ x - 1\,\big)\,\overline{2x^2 - 3x + 0} \\ -(2x^2 - 2x) \\ \hline -x + 0 \\ -(-x + 1) \\ \hline -1 \end{array}$$

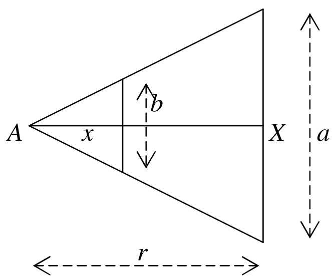
.: $\dfrac{x(2x - 3)}{x - 1} = 2x - 1 - \dfrac{1}{x - 1}$

As $x \to \pm\infty$, $y = \dfrac{x(2x - 3)}{x - 1} \to 2x - 1$

.: the equation of line ℓ is $y = 2x - 1$.

Q14c





Base width b of rectangle: $\dfrac{b}{a} = \dfrac{x}{r}$ (similar triangles), $b = \dfrac{ax}{r}$

Area of rectangle $= \dfrac{ax}{r}\sqrt{r^2 - x^2}$

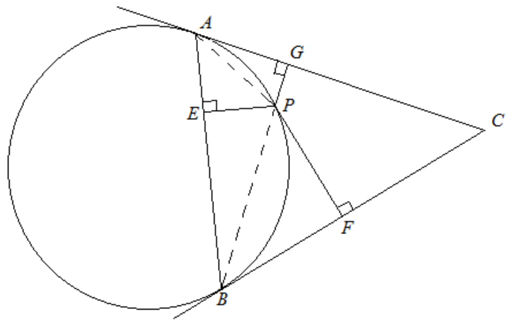
Volume of solid $ABCD = \int_0^r \dfrac{a}{r}x\sqrt{r^2 - x^2}\,dx$

$= -\dfrac{a}{2r}\int_{r^2}^{0}\sqrt{u}\,du$

$= \dfrac{a}{2r}\int_0^{r^2}\sqrt{u}\,du = \dfrac{a}{2r}\left[\dfrac{2u^{\frac{3}{2}}}{3}\right]_0^{r^2} = \dfrac{ar^2}{3}$

Let $u = r^2 - x^2$, $-\dfrac{1}{2}\dfrac{du}{dx} = x$, $x = 0$, $u = r^2$, $x = r$, $u = 0$

Q14di



$\angle PAG = \angle PBE$ (angles in alternate segments are equal)
$\angle PGA = \angle PEB = 90°$
.: ΔAPG and ΔBPE are similar.

Q14dii It can be shown that ΔAPE and ΔBPF are also similar using the same ideas as in Q14di.

.: $\frac{EP}{AP} = \frac{FP}{PB}$. From Q14di, $\frac{EP}{PB} = \frac{GP}{AP}$.

.: $\frac{PB}{AP} = \frac{FP}{EP} = \frac{EP}{GP}$, .: $EP^2 = FP \times GP$

Q15ai Without loss of generality, let $a \ge b \ge 0$, .: $a - b \ge 0$

$(a-b)^2 \ge 0$, $(a-b)^2 + 4ab \ge 4ab$, $(a+b)^2 \ge 4ab$

.: $a + b \ge 2\sqrt{ab}$, .: $\sqrt{ab} \le \frac{a+b}{2}$

Q15aii Given $y \ge x$ and $x \ge 1$

.: $y - x \ge 0$ and $x - 1 \ge 0$

.: $(y-x)(x-1) \ge 0$, .: $xy - x^2 - y + x \ge 0$, $xy - x^2 + x \ge y$

.: $x(y - x + 1) \ge y$

Q15aiii From Q15ai, $\sqrt{ab} \le \frac{a+b}{2}$. Let $a = j$ and $b = n - j + 1$.

.: $\sqrt{j(n-j+1)} \le \frac{n+1}{2}$

From Q15aii, $y \le x(y - x + 1)$. Let $y = n$ and $x = j$.

.: $n \le j(n - j + 1)$ and $\sqrt{n} \le \sqrt{j(n-j+1)}$

Hence $\sqrt{n} \le \sqrt{j(n-j+1)} \le \frac{n+1}{2}$.

Q15aiv For j satisfying $1 \le j \le n$, $\sqrt{n} \le \sqrt{j(n-j+1)} \le \frac{n+1}{2}$ is independent of j. Let $j = 1, 2, 3, \ldots, n$.

.: $\left(\sqrt{n}\right)^n \le \sqrt{1.n}\sqrt{2(n-1)}\sqrt{3(n-2)}\ldots\sqrt{n.1} \le \left(\frac{n+1}{2}\right)^n$

.: $\left(\sqrt{n}\right)^n \le \sqrt{(n!)^2} \le \left(\frac{n+1}{2}\right)^n$, .: $\left(\sqrt{n}\right)^n \le n! \le \left(\frac{n+1}{2}\right)^n$

Q15bi $P(z)$ has real coefficients, .: the conjugates of α and $i\alpha$, i.e. $\overline{\alpha}$ and $\overline{i\alpha}$, are also zeros of $P(z)$.

$\overline{i\alpha} = \overline{i}\,\overline{\alpha} = -i\overline{\alpha}$

Q15bii $P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$

$= \left(z^4 - 2kz^3 + k^2z^2\right) + \left(k^2z^2 - 2kz + 1\right)$

$= z^2\left(z^2 - 2kz + k^2\right) + \left(k^2z^2 - 2kz + 1\right) = z^2(z-k)^2 + (kz-1)^2$

Q15biii α, $\overline{\alpha}$, $i\alpha$ and $-i\overline{\alpha}$ are the zeros of $P(z)$.

.: $P(z) = (z-\alpha)(z-\overline{\alpha})(z-i\alpha)(z+i\overline{\alpha})$

If α is a real zero, $\overline{\alpha} = \alpha$, $P(z) = \left(z^2 + \alpha\overline{\alpha}\right)(z-\alpha)^2$

.: $P(z) = z^2(z-\alpha)^2 + \alpha\overline{\alpha}(z-\alpha)^2$

Compare with

$P(z) = z^2(z-k)^2 + (kz-1)^2 = z^2(z-k)^2 + k^2\left(z - \frac{1}{k}\right)^2$

$\alpha = k$, $\alpha\overline{\alpha} = k^2$ and $\alpha = \frac{1}{k}$

.: $k = \frac{1}{k}$, $k^2 = 1$, $k = \pm 1$, .: $\alpha = \pm 1$ and $\alpha\overline{\alpha} = 1$

.: $P(z) = \left(z^2 + 1\right)(z \pm 1)^2$

Q15biv $P(z) = (z-\alpha)(z-\overline{\alpha})(z-i\alpha)(z+i\overline{\alpha})$

$= z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$

.: $\alpha\overline{\alpha}(i\alpha)(-i\overline{\alpha}) = |\alpha|^4 = 1$, .: $|\alpha| = 1$

Q15bv $\alpha + \overline{\alpha} + i\alpha - i\overline{\alpha} = 2k$ where $\alpha = x + iy$

.: $2k = 2x - 2y$, $k = x - y$

Q15bvi $|\alpha| = 1$, $\alpha = x + iy = \cos\theta + i\sin\theta$

.: $k = x - y = \cos\theta - \sin\theta = A\cos(\theta + \phi)$

$= A\cos\theta\cos\phi - A\sin\theta\sin\phi$ where $A > 0$

.: $A\cos\phi = 1$, $A\sin\phi = 1$, .: $A = \sqrt{2}$ and $k = \sqrt{2}\cos(\theta + \phi)$

k is sinusoidal with amplitude of $\sqrt{2}$, .: $-\sqrt{2} \le k \le \sqrt{2}$

Q16ai Number of arrangements $= \frac{(m+n)!}{m!n!}$

Q16aii The boxes are different and there is no restriction on the number of coins in each box. This question can be considered as a partition problem of 10 identical coins in a row with 3 identical partitions.

○○○○○○○○○○|||

The example shown has the 10 coins in Box 1. Box 2, Box 3 and Box 4 are empty.

Number of ways $= \frac{13!}{10!3!} = 286$

Q16bi $\tan\left(\tan^{-1}x+\tan^{-1}y\right)=\dfrac{\tan\left(\tan^{-1}x\right)+\tan\left(\tan^{-1}y\right)}{1-\tan\left(\tan^{-1}x\right)\times\tan\left(\tan^{-1}y\right)}$

$=\dfrac{x+y}{1-xy}$, ∴ $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

Q16bii Prove $\displaystyle\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{n}{n+1}\right)$, $n\ge 1$

$n=1$, $\displaystyle\sum_{j=1}^{1}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{1}{1+1}\right)$, both sides $=\tan^{-1}\left(\frac{1}{2}\right)$

Assume it is true for $n=k$, i.e. $\displaystyle\sum_{j=1}^{k}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{k}{k+1}\right)$,

prove that it is also true for $n=k+1$,

i.e. $\displaystyle\sum_{j=1}^{k+1}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{(k+1)}{(k+1)+1}\right)$.

$$LHS=\sum_{j=1}^{k+1}\tan^{-1}\left(\frac{1}{2j^2}\right)=\sum_{j=1}^{k}\tan^{-1}\left(\frac{1}{2j^2}\right)+\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

$$=\tan^{-1}\left(\frac{k}{k+1}\right)+\tan^{-1}\left(\frac{1}{2(k+1)^2}\right)=\tan^{-1}\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{k+1}\times\frac{1}{2(k+1)^2}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{2k(k+1)+1}{2(k+1)^2}}{\frac{2(k+1)^3-k}{2(k+1)^3}}\right)=\tan^{-1}\frac{(k+1)(2k(k+1)+1)}{2(k+1)^3-k}$$

$$=\tan^{-1}\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}=\tan^{-1}\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}$$

$$=\tan^{-1}\left(\frac{(k+1)}{(k+1)+1}\right)=RHS$$

∴ $\displaystyle\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)=\tan^{-1}\left(\frac{n}{n+1}\right)$ for all positive integers *n*.

Q16biii $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right)=\lim_{n\to\infty}\tan^{-1}\left(\frac{n}{n+1}\right)$

$\displaystyle=\lim_{n\to\infty}\tan^{-1}\left(\frac{1}{1+\frac{1}{n}}\right)\to\tan^{-1}(1)=\frac{\pi}{4}$

Q16ci Number of ways in selecting *k* integers in different orders from *n* integers $={}^nP_k$; number of ways that one of the *k* integers is selected for the second time in the $(k+1)$th selection $={}^kC_1$; number of permutations of $k+1$ integers from *n* integers with repetition $=n^{k+1}$.

∴ $P(k)=\dfrac{{}^nP_k\times{}^kC_1}{n^{k+1}}=\dfrac{n!k}{(n-k)!\,n^{k+1}}=\dfrac{(n-1)!k}{n^k(n-k)!}$

Q16cii If $P(k)\ge P(k-1)$, then $\dfrac{(n-1)!k}{n^k(n-k)!}\ge\dfrac{(n-1)!(k-1)}{n^{k-1}(n-k+1)!}$

∴ $\dfrac{k}{n}\ge\dfrac{k-1}{n-k+1}$.

Since $1\le k\le n$, $k(n-k+1)\ge n(k-1)$, ∴ $k^2-k-n\le 0$.

Q16ciii If $\sqrt{n+\frac{1}{4}}>k-\frac{1}{2}$, then $n+\frac{1}{4}>k^2-k+\frac{1}{4}$,

∴ $n>k^2-k$. ∴ $n\ge k^2-k+1$, since *n* and *k* are integers.

∴ $n\ge k^2-k+\frac{1}{4}+\frac{3}{4}$, $n\ge\left(k-\frac{1}{2}\right)^2+\frac{3}{4}$

∴ $n>\left(k-\frac{1}{2}\right)^2$, $\sqrt{n}>k-\frac{1}{2}$

Q16civ If $P(k)\ge P(k-1)$, then $k^2-k-n\le 0$ where $1\le k\le n$.

∴ $1\le k\le\dfrac{1+\sqrt{4n+1}}{2}$ by the quadratic formula.

If $4n+1$ is not a perfect square, then $\dfrac{1+\sqrt{4n+1}}{2}$ is not an integer and $k<\dfrac{1+\sqrt{4n+1}}{2}$.

∴ $k<\dfrac{1+2\sqrt{n+\frac{1}{4}}}{2}$, i.e. $k<\frac{1}{2}+\sqrt{n+\frac{1}{4}}$.

Hence $k<\frac{1}{2}+\sqrt{n}$ from Q16ciii.

If $P(k)<P(k-1)$ or $P(k+1)<P(k)$, then $k^2-k-n>0$,

∴ $\dfrac{1+\sqrt{4n+1}}{2}<k\le n$, i.e. $\frac{1}{2}+\sqrt{n+\frac{1}{4}}<k\le n$.

Hence $P(k)$ is greatest when $k<\frac{1}{2}+\sqrt{n}$ and *k* is the closest integer to $\frac{1}{2}+\sqrt{n}$, provided $4n+1$ is not a perfect square.

When $k<\frac{1}{2}+\sqrt{n}$ and *k* is the closest integer to $\frac{1}{2}+\sqrt{n}$, *k* is also the closest integer to $\sqrt{n}$.

Note: If $4n+1$ is a perfect square, $P(k)$ is greatest when $k=\frac{1}{2}+\sqrt{n+\frac{1}{4}}$, e.g. when $n=2$, $k=2$ and $P(2)=P(1)$ is the greatest value; when $n=6$, $k=3$ and $P(3)=P(2)$ etc.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.