2012 NSW BOS Mathematics Exam Solutions

Section I

1	2	3	4	5	6	7	8	9	10
B	D	C	A	C	D	C	A	B	B

Q1 B

Q2 $\frac{1}{2\sqrt{5}-\sqrt{3}} = \frac{1}{2\sqrt{5}-\sqrt{3}} \times \frac{2\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}} = \frac{2\sqrt{5}+\sqrt{3}}{17}$ D

Q3 $x^2+3x-1=0$ has roots α and β,
$\therefore\ \alpha\beta=-1$, $\alpha+\beta=-3$
$\therefore\ \alpha\beta+(\alpha+\beta)=-1-3=-4$ C

Q4 At $x=a$, the gradient of the curve is *positive*, and it is *decreasing* as x increases, $\therefore\ f'(a)>0$ and $f''(a)<0$ A

Q5 Let (x, y) be a point on the line $y=3x+1$,
$\therefore\ (x, y)$ is $(x, 3x+1)$
Distance ℓ *between* (x, y) *and* $(2,-1)$
$= \sqrt{(x-2)^2+(3x+1-(-1))^2} = \sqrt{10x^2+8x+8}$
Let $\frac{d\ell}{dx} = \frac{20x+8}{2\sqrt{10x^2+8x+8}} = 0$, $\therefore\ x=-\frac{2}{5}$
$\therefore$ *perpendicular distance* $= \sqrt{10\left(-\frac{2}{5}\right)^2+8\left(-\frac{2}{5}\right)+8} = \frac{8}{\sqrt{10}}$ C

Alternative method: (1) Find the equation of the line perpendicular to $y=3x+1$. (2) Find the point of intersection of the two lines. (3) Find the distance between $(2,-1)$ and the point of intersection.

Q6 $\sqrt{3}\tan x=-1$, $0\le x\le 2\pi$
$\tan x=-\frac{1}{\sqrt{3}}$, $x=\frac{5\pi}{6}, \frac{11\pi}{6}$ D

Q7 $a=e^x$, $a^2=(e^x)^2=e^{2x}$, $\log_e(a^2)=2x$ C

Q8 Pick a point in the shaded region, say $(3,0)$, to test the inequalities. A

Q9 $\int_1^4 \frac{1}{3x}dx = \frac{1}{3}\int_1^4 \frac{1}{x}dx = \frac{1}{3}[\ln x]_1^4 = \frac{1}{3}\ln 4$ B

Q10 The values of $\int_2^7 f(x)dx$ and $\int_2^8 f(x)dx$ are negative.
$\therefore\ \int_0^2 f(x)dx$ has the greatest value. B

Section II

Q11a $2x^2-7x+3=(2x-1)(x-3)$

Q11b $|3x-1|<2$
$(3x-1)^2<4$, $9x^2-6x+1<4$
$\therefore\ 9x^2-6x-3<0$, $3x^2-2x-1<0$, $(3x+1)(x-1)<0$
$\left(x+\frac{1}{3}\right)(x-1)<0$, $\therefore\ -\frac{1}{3}<x<1$

Alternative method:
$3x-1<2$ when $3x-1\ge 0$ and $-(3x-1)<2$ when $3x-1<0$
$\therefore\ x<1$ when $x\ge\frac{1}{3}$ and $x>-\frac{1}{3}$ when $x<\frac{1}{3}$
$\therefore\ -\frac{1}{3}<x<1$

Q11c At $x=3$, $y=x^2=9$, $\frac{dy}{dx}=2x=6$
Equation of the tangent: $y-y_1=m(x-x_1)$
$y-9=6(x-3)$, $\therefore\ y=6x-9$

Q11d $\frac{d}{dx}\left(3+e^{2x}\right)^5 = 5\left(3+e^{2x}\right)^4\left(2e^{2x}\right) = 10e^{2x}\left(3+e^{2x}\right)^4$

Q11e $x^2=16(y-2)=4\times 4(y-2)$
The y-coordinate of the focus is given by $y-2=4$, i.e. $y=6$.
The point $(0,6)$ is the focus.

Q11f $A=\frac{1}{2}r^2\theta$, $\therefore\ A=\frac{1}{2}r\ell$ where $\ell=r\theta$ is the length of the arc, $\therefore\ 50=\frac{1}{2}\times 6\times\ell$, $\ell=\frac{50}{3}$ cm

Q11g $\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{2}dx = 2\left[\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} = 2\tan\frac{\pi}{4} = 2$

Q12ai $\frac{d}{dx}((x-1)\log_e x) = \left(\frac{d}{dx}(x-1)\right)(\log_e x)+(x-1)\frac{d}{dx}\log_e x$
$= \log_e x+(x-1)\times\frac{1}{x} = \log_e x+\frac{x-1}{x}$

Q12aii $\frac{d}{dx}\left(\frac{\cos x}{x^2}\right) = \frac{\left(x^2\right)\left(\frac{d}{dx}\cos x\right)-(\cos x)\left(\frac{d}{dx}x^2\right)}{x^4}$
$= \frac{x^2(-\sin x)-2x\cos x}{x^4} = -\frac{x(x\sin x+2\cos x)}{x^4} = -\frac{x\sin x+2\cos x}{x^3}$

Q12b $\int\frac{4x}{x^2+6}dx = 2\int\frac{2x}{x^2+6}dx = 2\int\frac{1}{u}du$
$= 2\ln u+c = 2\ln\left(x^2+6\right)+c$

Let $u=x^2+6$, $\frac{du}{dx}=2x$

Q12ci Arithmetic sequence: $a = 3$, $d = 2$
$t_{20} = 3 + (20-1)\times 2 = 41$

Q12cii $S_{20} = \frac{20}{2}(3+41) = 440$

Q12ciii $S_n = \frac{n}{2}(2\times 3 + (n-1)\times 2) \le 200$
$n(6+2n-2) \le 400$, $2n^2 + 4n \le 400$, $n^2 + 2n + 1 \le 201$
$(n+1)^2 \le 201$, $n+1 \le \sqrt{201}$, $n \le \sqrt{201} - 1$
$\therefore\ n = 13$

Q12di $Area \approx 3(0.5 + 4\times 2.3 + 2\times 2.9 + 4\times 3.8 + 2.1) = 98.4\ \text{m}^2$

Q12dii $Volume \approx 98.4\times 0.4\times 10 = 393.6\ \text{m}^3$

Q13ai $A(4,0)$, $B(0,8)$, $\therefore\ AB = \sqrt{(0-4)^2 + (8-0)^2} = 4\sqrt{5}$

Q13aii The cosine rule: $b^2 = c^2 + a^2 - 2ca\cos\angle ABC$
$5^2 = \left(4\sqrt{5}\right)^2 + \left(\sqrt{65}\right)^2 - 2\left(4\sqrt{5}\right)\left(\sqrt{65}\right)\cos\angle ABC$
$25 = 80 + 65 - 8\sqrt{325}\cos\angle ABC$, $\cos\angle ABC = \frac{80+65-25}{8\sqrt{325}}$,
$\therefore\ \angle ABC \approx 34°$

Q13aiii Let $N(p,q)$. $\therefore\ 2p + q = 8$ …… (1)
$m_{CN}\times m_{AB} = -1$, $\therefore\ \frac{q-4}{p-7}\times -2 = -1$ …… (2)
From (1), $q = 8 - 2p$ …… (3)
Substitute (3) in (2), $\frac{8-2p-4}{p-7}\times 2 = 1$, $\therefore\ 8 - 4p = p - 7$,
$\therefore\ p = 3$ and $q = 2$, $\therefore\ N(3,2)$

Q13bi Solve $y = x^2 - 3x$ and $y = 5x - x^2$ to find x, $x > 0$.
$x^2 - 3x = 5x - x^2$, $\therefore\ 2x^2 - 8x = 0$, $2x(x-4) = 0$
$\therefore\ x = 4$ is the x-coordinate of point A.

Q13bii $\textit{Area of the shaded region} = \int_0^4 \left[\left(5x - x^2\right) - \left(x^2 - 3x\right)\right]dx$
$= \int_0^4 \left(8x - 2x^2\right)dx = \left[4x^2 - \frac{2x^3}{3}\right]_0^4 = \frac{64}{3}$

Q13ci $\Pr(RR) = \frac{3}{5}\times\frac{3}{7} = \frac{9}{35}$

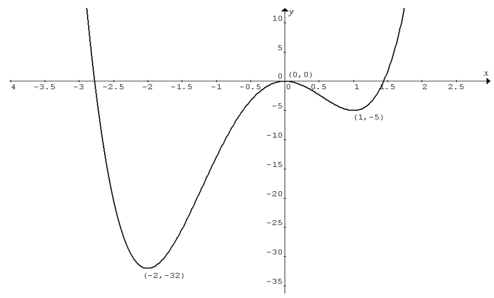
Q13cii $\Pr(at.least.one.white) = 1 - \Pr(RR) = 1 - \frac{9}{35} = \frac{26}{35}$

Q13ciii $\Pr(same.colour) = \Pr(RR) + \Pr(WW) = \frac{9}{35} + \frac{2}{5}\times\frac{4}{7} = \frac{17}{35}$

Q14ai $f(x) = 3x^4 + 4x^3 - 12x^2$
$f'(x) = 12x^3 + 12x^2 - 24x = 12x\left(x^2 + x - 2\right) = 12x(x-1)(x+2)$
Let $f'(x) = 0$, $12x(x-1)(x+2) = 0$
$\therefore\ x = -2$, 0, 1 and the corresponding $y = -32$, 0, -5.
The stationary points are $(-2,-32)$, $(0,0)$ and $(1,-5)$.

x		–2		0		1	
$f'(x)$	–	0	+	0	–	0	+
Nature		local min		local max		local min	

Q14aii



Q14aiii The function is increasing in the intervals $-2 < x < 0$ and $x > 1$.

Q14aiv $3x^4 + 4x^3 - 12x^2 + k = 0$ will have no solution if $k > 32$.

Q14b $Volume = \int_0^1 \pi\left(\frac{3}{(x+2)^2}\right)^2 dx = \int_0^1 \pi\frac{9}{(x+2)^4}dx$
$= 9\pi\int_0^1 (x+2)^{-4}dx = 9\pi\left[-\frac{1}{3(x+2)^3}\right]_0^1 = 9\pi\left(-\frac{1}{81} + \frac{1}{24}\right)$
$= \frac{19\pi}{72}$.

Q14ci $N(t) = 1000e^{kt}$, when $t = 20$, $N = 2000$
$\therefore\ 1000e^{20k} = 2000$, $e^{20k} = 2$, $k = \frac{\ln 2}{20} \approx 0.0347$

Q14cii $N(120) = 1000e^{120k} = 1000\left(e^{20k}\right)^6 = 1000\times 2^6 = 64000$

Q14ciii $\frac{dN}{dt} = kN(t)$
When $t = 120$, $\frac{dN}{dt} = kN(120) = \frac{\ln 2}{20}\times 64000 = 3200\ln 2$
≈ 2218 per minute

Q14civ When $N = 100000$, $1000\left(e^{20k}\right)^{\frac{t}{20}} = 100000$,
$\therefore\ 2^{\frac{t}{20}} = 100$, $\frac{t}{20} = \frac{\log_{10}100}{\log_{10}2}$, $t = \frac{40}{\log_{10}2} \approx 133$ minutes

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Q15ai Geometric series: $a = 10$, $r = 96\% = 0.96$

$$S_{10} = \frac{10\left(1-0.96^{10}\right)}{1-0.96} \approx 83.8 \text{ cm}$$

The length required is 83.8 cm approximately.

Q15aii $S_\infty = \frac{10}{1-0.96} = 250$ cm or 2.5 m

.: a 3-m strip is sufficient to make any number of rectangles.

Q15bi Assume that 'displacement' means displacement from the origin or simply position (relative to the origin).

$\dot{x} = 1 - 2\cos t$, at $t = 0$, $\dot{x} = 1 - 2\cos 0 = -1$

The initial velocity is 1 m s^{-1} to the left towards the origin.

Q15bii The velocity is maximum when $\cos t = -1$, $\dot{x} = 3$ m s^{-1}.

Q15biii $x = \int (1 - 2\cos t)dt = t - 2\sin t + c$

At $t = 0$, $x = 3$, .: $c = 3$.

.: $x = t - 2\sin t + 3$

Q15biv When $\dot{x} = 1 - 2\cos t = 0$, $\cos t = \frac{1}{2}$, $t = \frac{\pi}{3}$.

$$x = \frac{\pi}{3} - 2\sin\frac{\pi}{3} + 3 = \frac{\pi}{3} - \sqrt{3} + 3 \approx 2.32$$

Q15ci

$$\$A_1 = \$360000\left(1 + \frac{6}{12\times 100}\right) - \$M = \$(360000\times 1.005 - M)$$

.: $\$A_2 = \$(360000\times 1.005 - M)\times 1.005 - \$M$

$= \$\left(360000\times 1.005^2 - 1.005M - M\right)$

or $\$A_2 = \$\left(360000\times 1.005^2 - \frac{M\left(1.005^2 - 1\right)}{1.005 - 1}\right)$ by formula.

Q15cii $A_{300} = 360000\times 1.005^{300} - \frac{M\left(1.005^{300} - 1\right)}{1.005 - 1} = 0$

$$M = \frac{360000\times 1.005^{300}\times 0.005}{1.005^{300} - 1} \approx 2319.485 \approx 2319.50 \text{ dollars}$$

Q15ciii $\$A_n < \$180000$,

$$360000\times 1.005^n - \frac{2319.485\left(1.005^n - 1\right)}{1.005 - 1} < 180000$$

$$360000\times 1.005^n - 463897.009\left(1.005^n - 1\right) < 180000$$

$103897\times 1.005^n > 283897$, $1.005^n > 2.732485$

$n > \frac{\ln 2.732485}{\ln 1.005} = 201.54$, .: after 202 months.

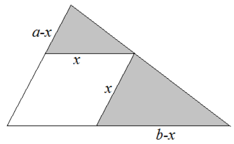
Q16ai $\angle EBF = \angle AED$, corresponding angles are equal

$\angle BEF = \angle EAD$, corresponding angles are equal

.: $\angle BFE = \angle EDA$

Hence ΔEBF and ΔAED are similar.

Q16aii



$\frac{a-x}{x} = \frac{x}{b-x}$, .: $x^2 = (a-x)(b-x)$, $x^2 = ab - (a+b)x + x^2$

$(a+b)x = ab$, .: $x = \frac{ab}{a+b}$

Q16bi $m_{OT} = \tan\theta = \frac{\sin\theta}{\cos\theta}$, .: $m_{PT} = -\frac{1}{m_{OT}} = -\frac{\cos\theta}{\sin\theta}$

On the unit circle, point $T(\cos\theta, \sin\theta)$.

Equation of tangent PT:

$y - \sin\theta = -\frac{\cos\theta}{\sin\theta}(x - \cos\theta)$, $x\cos\theta + y\sin\theta = \sin^2\theta + \cos^2\theta$

.: $x\cos\theta + y\sin\theta = 1$

Q16bii $x\cos\theta + y\sin\theta = 1$

When $y = 1$, $x = \frac{1-\sin\theta}{\cos\theta}$, .: $Q\left(\frac{1-\sin\theta}{\cos\theta}, 1\right)$, .: $BQ = \frac{1-\sin\theta}{\cos\theta}$.

Q16biii $x\cos\theta + y\sin\theta = 1$

When $y = 0$, $x = \frac{1}{\cos\theta}$, .: $P\left(\frac{1}{\cos\theta}, 0\right)$, .: $OP = \frac{1}{\cos\theta}$.

Area of trapezium $A = \frac{1}{2}\left(\frac{1-\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\times 1 = \frac{2-\sin\theta}{2\cos\theta}$.

Q16biv $\frac{dA}{d\theta} = \frac{(2\cos\theta)(-\cos\theta) - (2-\sin\theta)(-2\sin\theta)}{4\cos^2\theta}$

$= \frac{2\sin\theta - 1}{2\cos^2\theta}$. For minimum area, let $\frac{dA}{d\theta} = 0$, $\frac{2\sin\theta - 1}{2\cos^2\theta} = 0$,

.: $2\sin\theta - 1 = 0$, $\sin\theta = \frac{1}{2}$, .: $\theta = \frac{\pi}{6}$.

Q16ci Solve $x^2 + (y-c)^2 = r^2$ and $y = x^2$ simultaneously to find the y-coordinate of the points of touching.

By substitution, $y + (y-c)^2 = r^2$

Expand and simplify, $y^2 + (1-2c)y + \left(c^2 - r^2\right) = 0$

Solve for y using the quadratic formula and simplify,

$$y = \frac{2c - 1 \pm \sqrt{1 - 4c + 4r^2}}{2}.$$

Since the two points have the same y-coordinate,

.: $1 - 4c + 4r^2 = 0$, .: $4c = 1 + 4r^2$

Q16cii From part ci, and $y > 0$, .: $y = \frac{2c-1}{2} > 0$, .: $c > \frac{1}{2}$.

Please inform mathline@itute.com re conceptual and/or mathematical errors.