2012 VCAA Math. Methods Exam 2 Solutions

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Note: Some steps can be done by CAS to save time

SECTION 1

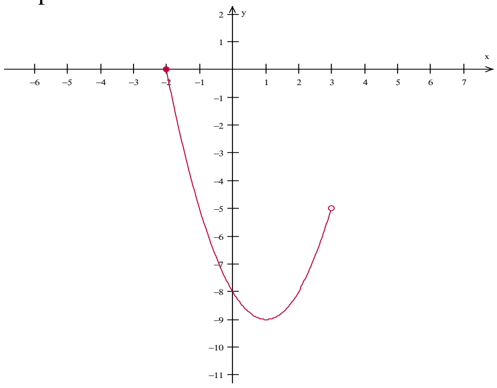
1	2	3	4	5	6	7	8	9	10	11
C	E	D	A	B	C	D	A	D	C	E

12	13	14	15	16	17	18	19	20	21	22
A	B	D	B	D	B	A	B	E	A	B

Q1 $T=\dfrac{2\pi}{\frac{\pi}{5}}=10$ C

Q2 *Average rate of change* $=\dfrac{f(3)-f(1)}{3-1}=9$ E

Q3 See Graph below. D

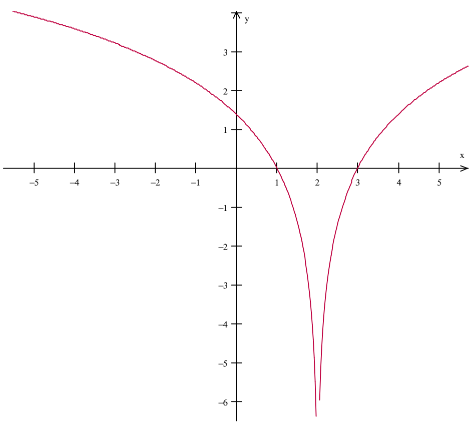


Q4 Let $u=e^{kx}$, $y=g(u)$

$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=g'(u)\times ke^{kx}=kg'\left(e^{kx}\right)e^{kx}$ A

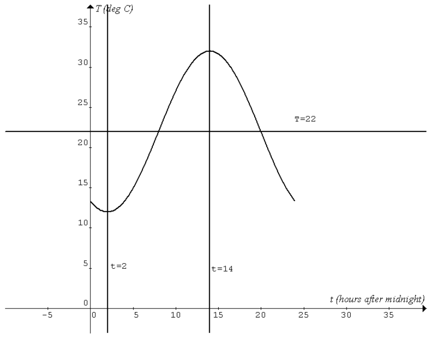
Q5 $g(x)=\log_e\left((x-2)^2\right)$, $(x-2)^2>0$, .: $x\neq 2$

See graph below. B

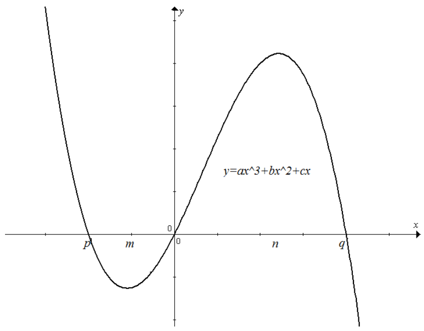


Q6 $T=\dfrac{\pi}{n}=\dfrac{\pi}{2}$, .: $n=2$. There is also a translation of $\dfrac{\pi}{4}$ in the positive x-direction. C

Q7 See Graph below. D



Q8 See graph below. A



Q9 $y=\sqrt{b-x^2}$, $\dfrac{dy}{dx}=-\dfrac{x}{\sqrt{b-x^2}}$

At $x=1$, $m_t=\dfrac{1}{\sqrt{b-1}}$

.: $m_n=-\dfrac{1}{m_t}=\sqrt{b-1}=3$, .: $b=10$ D

Q10 Solve $\dfrac{\int_0^a \sin^2 x\,dx}{a-0}=0.4$, $\left[x-\dfrac{\sin 2x}{2}\right]_0^a=0.4a$,

$a-\dfrac{\sin 2a}{2}=0.4a$, $a=1.298$ C

Q11 Normal distribution: $\mu=252$, $\sigma=12$

$\Pr(X>x)=0.40$, $\Pr(X<x)=0.60$,

$\text{invNorm}(0.60,252,12)\approx 255.0$ E

Q12 $W\to W(0.7)$, $W\to L(0.3)$, $L\to L(0.6)$, $L\to W(0.4)$

$\Pr(exactly.one.of.next.two)$

$=\Pr(WWL)+\Pr(WLW)=1\times0.7\times0.3+1\times0.3\times0.4=0.33$ A

Q13 $\Pr(B'\mid A)=\dfrac{\Pr(B'\cap A)}{\Pr(A)}=\dfrac{\Pr(B'\cap A)}{\Pr(B'\cap A)+\Pr(B\cap A)}$

$=\dfrac{\frac{3}{7}}{\frac{3}{7}+\frac{2}{5}}=\dfrac{15}{29}$ B

Q14 $y=\sqrt{x}=1$, $x=1$; $y=\sqrt{x}=2$, $x=4$; $y=\sqrt{x}=3$, $x=9$; $y=\sqrt{x}=4$, $x=16$

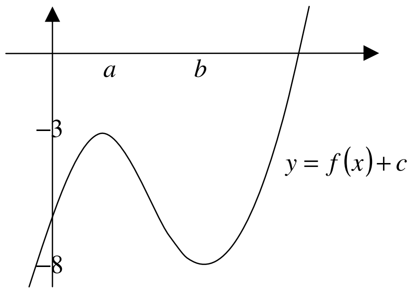
Total area $=1\times1+4\times1+9\times1+16\times1=30$ D

Q15 $f'(x)=3x^2-4$

$f(x)=\int(3x^2-4)dx=x^3-4x+c\ =x(x+2)(x-2)+c$

If $c=0$, $f(x)=x(x+2)(x-2)$ and there will be 3 x-intercepts at $x=-2$, $x=0$ and $x=2$. Choose an appropriate $c>0$ to obtain graph B. B

Q16



As c increases from 0, the graph moves up and the local maximum approaches the x-axis. When $c<3$, there is only 1 x-intercept. When $c=3$, the local maximum is at the x-axis and there are 2 x-intercepts. When $3<c<8$, there are 3 x-intercepts. When $c=8$, 2 x-intercepts. When $c>8$, 1 x-intercept. D

Q17 Change the matrix equation to two simultaneous linear equations, writing y as the subject of the equations.

$y=-\frac{m}{3}x+\frac{1}{3}$ and $y=-\frac{1}{m+2}x+\frac{m}{m+2}$

The equation will have no solution when:

$\frac{m}{3}=\frac{1}{m+2}$ AND $\frac{1}{3}\neq\frac{m}{m+2}$

i.e. $m^2+2m-3=0$, $(m-1)(m+3)=0$ AND $m\neq1$

$\therefore$ $m=-3$ B

Q18 $y=\log_e x$, $\frac{dy}{dx}=\frac{1}{x}$.

At $(a,\log_e a)$, $m_t=\frac{1}{a}$ and the equation of the tangent is:

$y-\log_e a=\frac{1}{a}(x-a)$.

When the tangent passes through $(0,0)$, $\log_e a=1$, $\therefore$ $a=e$ and $m_t=\frac{1}{e}$.

For $b<0$, $m_t<\frac{1}{e}$, i.e. $\frac{1}{a}<\frac{1}{e}$, $\therefore$ $a>e$ A

Q19 $f(\pi-\theta)=-f(\theta)$ and $f(\pi-\theta)=-f(-\theta)$

$\therefore$ $f(\theta)=f(-\theta)$, $\therefore$ $f(x)$ is even. Only B is even. B

Q20 $\Pr(X=k)=(1-p)^k p$

$\Pr(X>1)=1-\Pr(X=0)-\Pr(X=1)=1-p-(1-p)p$

$=(1-p)^2$ E

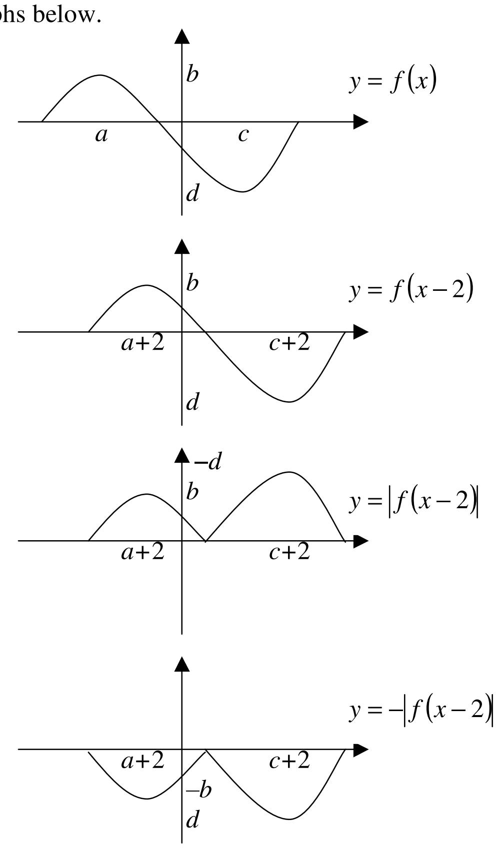
Q21 $V=\frac{1}{3}\pi h^3$, $\frac{dV}{dh}=\pi h^2$

$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $\frac{dV}{dt}=\pi h^2\times\frac{dh}{dt}$

When $h=5$, $-300=25\pi\times\frac{dh}{dt}$, $\frac{dh}{dt}=-\frac{12}{\pi}$

The rate of decrease is $\frac{12}{\pi}$. A

Q22 See graphs below. B



SECTION 2

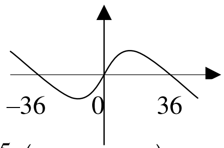
Q1a *Total surface area* $= 2x\left(\frac{5x}{2}\right) + 2hx + 2h\left(\frac{5x}{2}\right) = 6480$

$\therefore\ 7hx = 6480 - 5x^2$, $h = \frac{6480 - 5x^2}{7x}$ where $x > 0$.

Q1b $V(x) = \frac{5x\left(6480 - 5x^2\right)}{14} > 0$, $\therefore\ x\left(6480 - 5x^2\right) > 0$,

$x\left(1296 - x^2\right) > 0$, $x(36 - x)(36 + x) > 0$,

$\therefore\ 0 < x < 36$



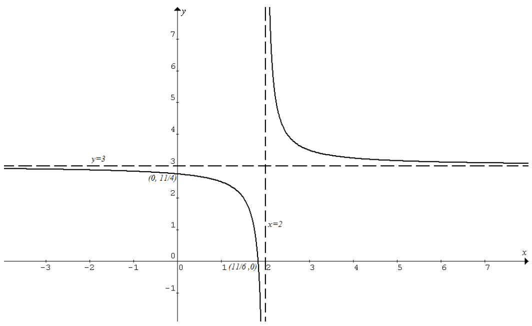
Q1c $\frac{dV}{dx} = \frac{5}{14}\left(6480 - 5x^2 + x(-10x)\right) = \frac{5}{14}\left(6480 - 15x^2\right)$

$= -\frac{75}{14}x^2 + \frac{16200}{7}$

Q1d For maximum volume, let $\frac{dV}{dx} = \frac{5}{14}\left(6480 - 15x^2\right) = 0$ where $x > 0$.

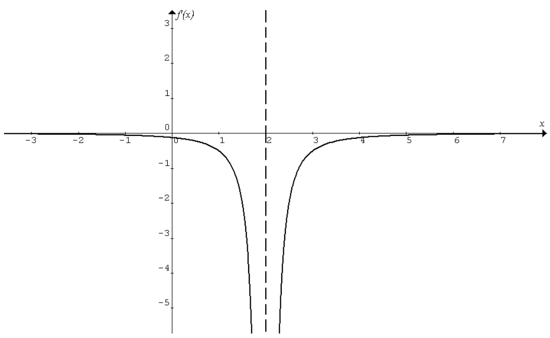
$\therefore\ x^2 = \frac{6480}{15}$, $x = \sqrt{432} = 12\sqrt{3}$ and $h = \frac{6480 - 5x^2}{7x} = \frac{120\sqrt{3}}{7}$.

Q2a



Q2bi By CAS, $f'(x) = -\frac{1}{2(x-2)^2}$ or $f'(x) = -\frac{2}{(2x-4)^2}$

Q2bii Sketch $y = f'(x)$ by CAS, the range is $(-\infty, 0)$.



Q2biii From part ii, $f'(x) < 0$, $\therefore\ f$ has no stationary points.

Q2c Given (p, q) is any point on $y = \frac{1}{2x-4} + 3$.

When $x = p$, $q = \frac{1}{2p-4} + 3$ and $f'(p) = -\frac{2}{(2p-4)^2}$.

Equation of the tangent at (p, q):

$y - \left(\frac{1}{2p-4} + 3\right) = -\frac{2}{(2p-4)^2}(x - p)$

$\therefore\ (2p-4)^2(y-3) - (2p-4) = -2(x-p)$

$(2p-4)^2(y-3) = -2x + 4p - 4$

Q2d $(2p-4)^2(y-3) = -2x + 4p - 4$

$\left(-1, \frac{7}{2}\right)$ lies on the tangents,

$\therefore\ (2p-4)^2\left(\frac{7}{2} - 3\right) = -2(-1) + 4p - 4$

Solve by CAS: $p = 1$ or 5

and $q = \frac{1}{2p-4} + 3 = \frac{5}{2}$ or $\frac{19}{6}$ respectively.

The two points are $\left(1, \frac{5}{2}\right)$ and $\left(5, \frac{19}{6}\right)$.

Q2e $f(x) = \frac{1}{2x-4} + 3 \rightarrow g(x) = \frac{1}{x}$

The required transformation sequence is:

(1) Dilate $f(x)$ by a factor of 2 horizontally,

i.e. $f(x) \rightarrow f\left(\frac{x}{2}\right) = \frac{1}{x-4} + 3$

(2) Translate to the left by 4 units,

i.e. $\rightarrow f\left(\frac{x+4}{2}\right) = \frac{1}{x} + 3$

(3) Translate downwards by 3 units,

i.e. $\rightarrow f\left(\frac{x+4}{2}\right) - 3 = \frac{1}{x} = g(x)$

$\therefore\ a = 2$, $c = -4$ and $d = -3$

Q3ai $\Pr(CCC) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$

Q3aii $\Pr(X \geq 10) =$ Binomial Cdf (20, ¼, 10, 20) ≈ 0.0139

Q3aiii $Var(X) = np(1-p)$, $\frac{75}{16} = n\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)$, $n = 25$

Q3bi $C \xrightarrow{\frac{3}{4}} C$, $C \xrightarrow{\frac{1}{4}} C'$, $C' \xrightarrow{\frac{2}{3}} C'$, $C' \xrightarrow{\frac{1}{3}} C$

There are two mutually exclusive chains: $C'CCCC$ and $C'C'CCC$

Probability $= 1 \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} + 1 \times \frac{2}{3} \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} = \frac{17}{64}$

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Q3bii The transition matrix is $\begin{matrix} & C & C' \\ & \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix} & \end{matrix} \begin{matrix} C \\ C' \end{matrix}$

The initial state matrix is $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} C \\ C' \end{matrix}$

$\begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix}^{24} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.5714 \\ \ \end{bmatrix} \begin{matrix} C \\ C' \end{matrix}$.: $\Pr(C_{25}) \approx 0.5714$

Q3c Random variable Y has a binomial distribution, $n = 25$.
$\Pr(Y > 23) = 6\Pr(Y = 25)$,
i.e. $\Pr(Y = 24) + \Pr(Y = 25) = 6\Pr(Y = 25)$
.: $\Pr(Y = 24) = 5\Pr(Y = 25)$, ${}^{25}C_{24}\,p^{24}(1-p) = 5p^{25}$
.: $25(1-p) = 5p$, $p = \frac{5}{6}$

Q3d $\Pr(Y \ge 18) = \Pr(W \ge 20)$, $\Pr(Y < 18) = \Pr(W < 20)$
.: $\Pr(Y \le 17) = \Pr\left(Z < \frac{20-a}{b}\right)$
.: $0.0447319277 \approx \Pr\left(Z < \frac{20-a}{b}\right)$, by CAS
.: $\frac{20-a}{b} \approx -1.698232677$ …… (1), by CAS
$\Pr(Y \ge 22) = \Pr(W \ge 25)$, $\Pr(Y < 22) = \Pr(W < 25)$
.: $\Pr(Y \le 21) = \Pr\left(Z < \frac{25-a}{b}\right)$
.: $0.618433506 \approx \Pr\left(Z < \frac{25-a}{b}\right)$, by CAS
.: $\frac{25-a}{b} \approx 0.3013691895$ …… (2), by CAS
Solve (1) and (2) simultaneously to find $a \approx 24.246$ and $b \approx 2.500$.

Q4ai $\frac{r}{h} = \frac{2}{10}$, $r = \frac{h}{5}$

Q4aii $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} \times \left(\frac{h}{5}\right)^2 h = \frac{\pi h^3}{75}$ m^3

Q4b $h = 10 + \frac{1}{1600}\left(t^3 - 1200t\right)$
When $t = 20$, $h = 10 + \frac{1}{1600}\left(20^3 - 1200 \times 20\right) = 0$,
.: $V = 0$, i.e. the tank is empty.

Q4ci When $t = 5$, $h = 10 + \frac{1}{1600}\left(5^3 - 1200 \times 5\right) = \frac{405}{64}$ m

Q4cii $h = 10 + \frac{1}{1600}\left(t^3 - 1200t\right)$, $\frac{dh}{dt} = \frac{1}{1600}\left(3t^2 - 1200\right)$

When $t = 5$, $h = \frac{405}{64}$ and $\frac{dh}{dt} = -\frac{45}{64}$
.: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{\pi h^2}{25} \times \frac{dh}{dt} = \frac{\pi}{25}\left(\frac{405}{64}\right)^2\left(-\frac{45}{64}\right) \approx -3.5$
.: the rate of decrease is 3.5 m^3 per minute.

Q4d When $h = 2$, $10 + \frac{1}{1600}\left(t^3 - 1200t\right) = 2$, $t = 12.2$ min

Q4e When $h = 2$, *volume of liquid* $= \frac{\pi h^3}{75} = \frac{8\pi}{75}$ m^3

Time required $= \frac{8\pi}{75} \div 0.2 = \frac{8\pi}{15}$ minutes

Q4f *Time available* $\approx (20 - 12.2) + \frac{8\pi}{15} \approx 9.5$ minutes

Q5ai $\int_{-2}^{0} e^x\,dx = \left[e^x\right]_{-2}^{0} = 1 - \frac{1}{e^2}$ or by CAS

Q5aii Same area as in part i.

Q5aiii *Total area of the shaded region*
$= \int_{-2}^{1} e^x\,dx = \left[e^x\right]_{-2}^{1} = e - \frac{1}{e^2}$

Q5bi $g(x) = k(x)$, $\log_e(x) = -\log_e(a - x)$,
$\log_e(x) + \log_e(a - x) = 0$, $\log_e x(a - x) = 0$,
.: $x(a - x) = 1$, $x^2 - ax + 1 = 0$.
The x-coordinates of the points of intersection are
$x_1 = \frac{a - \sqrt{a^2 - 4}}{2}$ and $x_2 = \frac{a + \sqrt{a^2 - 4}}{2}$, given $a > 0$.

Q5bii For distinct points of intersection, $a^2 - 4 > 0$, .: $a > 2$

Q5c The x-coordinate of the midpoint of AB
$= \frac{x_1 + x_2}{2} = \frac{a}{2} = \sqrt{2}$, .: $a = 2\sqrt{2}$

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