2013 NSW BOS Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	D	C	D	A	B	A	D	B	C

Q1 $P(x)=x^3-4x^2-6x+k$ and $x-2$ is a factor

.: $P(2)=2^3-4\times 2^2-6\times 2+k=0$, .: $k=20$ C

Q2 $y=f^{-1}(x)$ is the reflection of $y=f(x)$ in the line $y=x$. D

Q3 $\angle ABC=\frac{1}{2}\left(2\pi-\frac{3\pi}{5}\right)=\frac{7\pi}{10}$ C

Q4 $y=x(1-x)^3(3-x)^2=-x(x-1)^3(x-3)^2$

$(x-1)^3$ indicates a stationary point of inflection on the *x*-axis at $x=1$, and $(x-3)^2$ indicates a turning point on the *x*-axis at $x=3$. $y\to-\infty$ as $x\to+\infty$ due to the negative sign. D

Q5 $u=1+2x$, $x=\frac{1}{2}(u-1)$, $dx=\frac{1}{2}du$

$\int x\sqrt{1+2x}\,dx=\int\frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du=\frac{1}{4}\int(u-1)\sqrt{u}\,du$ A

Q6 $\sin 2x=a$, $2x=2k\pi+\sin^{-1}a$ or $(2k+1)\pi-\sin^{-1}a$

.: $2x=n\pi+(-1)^n\sin^{-1}a$ which gives the former if *n* is even; and the latter if *n* is odd.

.: $x=\frac{n\pi+(-1)^n\sin^{-1}a}{2}$ B

Q7 The youngest can sit at any one of the 8 seats, .: in 8 ways. The second youngest can sit on either side of the youngest, .: in 2 ways. The remaining 6 of the family can sit around the table in 6! ways. .: total number of ways is $8\times 2\times 6!$
Without restrictions the total number of ways is 8!

.: $\Pr(youngest2together)=\frac{8\times 2\times 6!}{8!}=\frac{2\times 6!}{7!}$ A

Q8 $\sin\theta=\frac{5}{13}$ and $\frac{\pi}{2}<\theta<\pi$, .: $\cos\theta=-\sqrt{1-\sin^2\theta}=-\frac{12}{13}$

.: $\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{5}{13}\times\frac{-12}{13}=-\frac{120}{169}$ D

Q9 The given diagram shows the graph of $y=\sin^{-1}x$ translated upwards by $\frac{\pi}{2}$ units. B

Q10 $|x+2|+|x-3|=\begin{cases}-2x+1 & when \quad x<-2\\ 5 & when \quad -2\le x\le 3\\ 2x-1 & when \quad x>3\end{cases}$

$x^2-x-6\le 0$, $(x+2)(x-3)\le 0$, .: $-2\le x\le 3$ C

Section II

Q11a $2x^3-3x^2-11x+7=2(x-\alpha)(x-\beta)(x-\gamma)$, .: $\alpha\beta\gamma=-\frac{7}{2}$

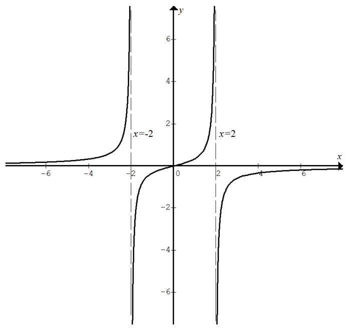
Q11b $\int\frac{1}{\sqrt{49-4x^2}}dx=\int\frac{1}{\sqrt{7^2-(2x)^2}}dx=\frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right)+c$

Q11c $n=10$, $p=\frac{1}{4}$, $\Pr(X=7)=\binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$

Q11di $f(x)=\frac{x}{4-x^2}$, $f'(x)=\frac{4-x^2-x(-2x)}{(4-x^2)^2}=\frac{4+x^2}{(4-x^2)^2}$

Since both $4+x^2$ and $(4-x^2)^2$ are positive, .: $f'(x)>0$ for all *x* in the domain of $f(x)$.

Q11dii $f(x)=\frac{x}{4-x^2}=-\frac{x}{(x+2)(x-2)}$



Q11e $\lim_{x\to 0}\frac{\sin\frac{x}{2}}{3x}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{6\times\frac{x}{2}}=\frac{1}{6}\times\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{1}{6}\times 1=\frac{1}{6}$

Q11f $u=e^{3x}$, $\frac{1}{3}du=e^{3x}dx$. When $x=0$, $u=1$; $x=\frac{1}{3}$, $u=e$.

$\int_0^{\frac{1}{3}}\frac{e^{3x}}{e^{6x}+1}dx=\frac{1}{3}\int_1^e\frac{du}{u^2+1}=\frac{1}{3}\left[\tan^{-1}u\right]_1^e=\frac{1}{3}\left(\tan^{-1}e-\frac{\pi}{4}\right)$

Q11g $\frac{d}{dx}\left(x^2\sin^{-1}5x\right)=\left(\sin^{-1}5x\right)(2x)+\left(x^2\right)\left(\frac{5}{\sqrt{1-(5x)^2}}\right)$

$=x\left(2\sin^{-1}5x+\frac{5x}{\sqrt{1-25x^2}}\right)$

Q12ai

$\sqrt{3}\cos x-\sin x=2\cos(x+\alpha)=2\cos x\cos\alpha-2\sin x\sin\alpha$

.: $\cos\alpha=\frac{\sqrt{3}}{2}$ and $\sin\alpha=\frac{1}{2}$, .: $\tan\alpha=\frac{1}{\sqrt{3}}$, $\alpha=\frac{\pi}{6}$

.: $\sqrt{3}\cos x-\sin x=2\cos\left(x+\frac{\pi}{6}\right)$

Q12aii $\sqrt{3}\cos x = 1 + \sin x$ and $0 < x < 2\pi$

.: $\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right) = 1$, .: $\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$

.: $x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$, .: $x = \frac{\pi}{6}, \frac{3\pi}{2}$

Q12b $Volume = \int_0^{\frac{3\pi}{2}} \pi y^2 dx = \int_0^{\frac{3\pi}{2}} \pi\left(3\sin\frac{x}{2}\right)^2 dx = \frac{9\pi}{2}\int_0^{\frac{3\pi}{2}} (1 - \cos x) dx$

$= \frac{9\pi}{2}\left[x - \sin x\right]_0^{\frac{3\pi}{2}} = \frac{9\pi}{2}\left(\frac{3\pi}{2} + 1\right)$ cubic units

Q12c $T = A + Be^{-kt}$

When $t = 0$, $T = 80$; $t = 10$, $T = 60$; $t \to \infty$, $T \to 22$.

.: $A + B = 80$, $A + Be^{-10k} = 60$ and $A = 22$

.: $B = 58$ and $e^{-10k} = \frac{19}{29}$, .: $k = \frac{1}{-10}\log_e\frac{19}{29} \approx 0.04229$

.: $T \approx 22 + 58e^{-0.04229t}$, .: $t \approx \frac{1}{-0.04229}\log_e\frac{T - 22}{58}$

When $T = 40$, $t \approx 28$ minutes

Q12di $D(t) = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ where (x_1, y_1) is $(t, t^2 + 3)$ and $ax + by + c = 0$ is $2x - y - 1 = 0$.

.: $D(t) = \frac{|2t - t^2 - 3 - 1|}{\sqrt{2^2 + (-1)^2}} = \frac{|2t - t^2 - 4|}{\sqrt{5}} = \frac{|t^2 - 2t + 4|}{\sqrt{5}} = \frac{t^2 - 2t + 4}{\sqrt{5}}$

since $t^2 - 2t + 4 = (t - 1)^2 + 3 > 0$ for all $t \in R$.

Q12dii Let $\frac{dD}{dt} = \frac{2t - 2}{\sqrt{5}} = 0$, .: $t = 1$ when *P* is closest to ℓ.

Q12diii $y = x^2 + 3$, when $t = 1$, $P(1,4)$, $m = \frac{dy}{dx} = 2x = 2(1) = 2$

.: the tangent at *P* is parallel to ℓ.

Q12e $v^2 + 9x^2 = k$, $\frac{1}{2}v^2 = \frac{k}{2} - \frac{9}{2}x^2$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x$

.: the particle moves in simple harmonic motion, and the period is $\frac{2\pi}{\sqrt{9}} = \frac{2\pi}{3}$

Q13ai Given $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$ and $\frac{dV}{dt} = -10^{-4}A$

.: $\frac{dV}{dr} = 4\pi r^2 = A$ and $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = -10^{-4}A$

.: $A \times \frac{dr}{dt} = -10^{-4}A$, .: $\frac{dr}{dt} = -10^{-4}$ is constant

Q13aii When $t = 0$, $V = \frac{4}{3}\pi r^3 = 10^{-6}\text{ m}^3$, .: $r \approx 0.0062\text{ m}$

Since $\frac{dr}{dt} = -10^{-4}$ is constant, .: *time taken* $\approx \frac{0.0062}{10^{-4}} \approx 62\text{ s}$

Q13bi Let $G(x, y)$ divides *NT* externally in the ratio $2:1$.

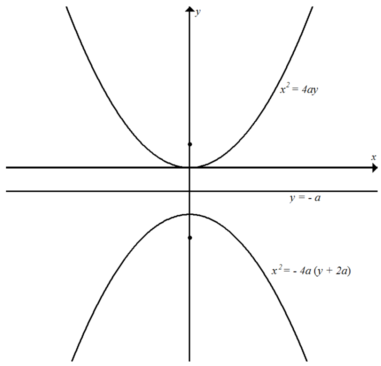
.: $T(ap, 0)$ is the midpoint of *NG*

.: $(ap, 0) = \left(\frac{0 + x}{2}, \frac{2a + ap^2 + y}{2}\right)$

.: $x = 2ap$ and $y = -2a - ap^2$, i.e. $G(2ap, -2a - ap^2)$

Q13bii $x = 2ap$, $p = \frac{x}{2a}$, .: $y = -2a - a\left(\frac{x}{2a}\right)^2$

.: $x^2 = -4a(y + 2a)$. It is a parabola with directrix $y = a - 2a = -a$ and focal length *a*, same directrix and focal length as the original parabola.



Q13ci Given $y = Vt\sin\theta - \frac{g}{2}t^2$, for projectile fired at *A*,

$y = ut\sin\alpha - \frac{g}{2}t^2$, $\frac{dy}{dt} = u\sin\alpha - gt$.

At maximum height, $\frac{dy}{dt} = u\sin\alpha - gt = 0$, .: $t = \frac{u\sin\alpha}{g}$

A13cii

For projectile fired at *B*, $y = wt\sin\beta - \frac{g}{2}t^2$, $\frac{dy}{dt} = w\sin\beta - gt$.

At maximum height, $\frac{dy}{dt} = w\sin\beta - gt = 0$, .: $t = \frac{w\sin\beta}{g}$.

Since they collide, .: they reach the same maximum height at the same time, .: $\frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$, .: $u\sin\alpha = w\sin\beta$

Q13ciii Given $x = Vt\cos\theta$, for projectile fired at *A*,

$x = u\left(\frac{u\sin\alpha}{g}\right)\cos\alpha = \frac{1}{g}(u\sin\alpha)(u\cos\alpha)$.

Similarly, for projectile fired at *B*, $x = \frac{1}{g}(w\sin\beta)(w\cos\beta)$.

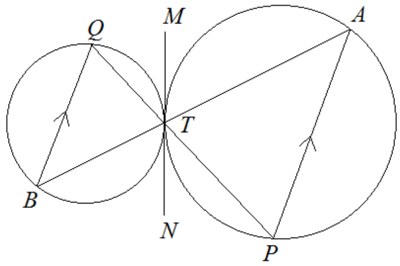
.: $d = \frac{1}{g}(w\sin\beta)(w\cos\beta) + \frac{1}{g}(u\sin\alpha)(u\cos\alpha)$

$= \frac{1}{g}\left((w\sin\beta)(w\cos\beta) + (u\sin\alpha)(u\cos\alpha)\right)$

$= \frac{1}{g}\left((u\sin\alpha)(w\cos\beta) + (w\sin\beta)(u\cos\alpha)\right)$

$= \frac{uw}{g}\sin(\alpha + \beta)$

Q13d Draw common tangent MN, chords QT and PT.



$\angle QTM = \angle QBT$ and $\angle PTN = \angle PAT$ (angles in alternate segments)
$\angle QBT = \angle PAT$ (alternate angles)
.: $\angle QTM = \angle PTN$, .: they are vertically opposite angles and .: QTP is a straight line segment, i.e. Q, T and P are collinear.

Q14ai

$$\frac{1}{(k+1)^2}-\frac{1}{k}+\frac{1}{k+1}=\frac{k-(k+1)^2+k(k+1)}{k(k+1)^2}=\frac{-1}{k(k+1)^2}<0$$

given $k>0$.

Q14aii For $n=2$, $\frac{1}{1^2}+\frac{1}{2^2}<2-\frac{1}{2}$ is true.

Assume the statement is true for $n=k$,

i.e. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}<2-\frac{1}{k}$

then

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

since $\frac{1}{(k+1)^2}-\frac{1}{k}<-\frac{1}{k+1}$ from Q14ai.

.: the statement is true for $n=k+1$
.: it is true for all integers $n\ge 2$.

Q14bi The term consisting of x^{2n} in the expansion of $(1+x)^{4n}$ is $\binom{4n}{2n}\left(1^{2n}\right)\left(x^{2n}\right)=\binom{4n}{2n}x^{2n}$, .: *coefficient* $=\binom{4n}{2n}$

Q14bii

$$\left(1+\left(x^2+2x\right)\right)^{2n}=\binom{2n}{0}1^0\left(x^2+2x\right)^{2n}+\binom{2n}{1}1^1\left(x^2+2x\right)^{2n-1}$$

$$+\binom{2n}{2}1^2\left(x^2+2x\right)^{2n-2}+...+\binom{2n}{2n}1^{2n}\left(x^2+2x\right)^0$$

$$=\sum_{k=0}^{2n}\binom{2n}{k}\left(x^2+2x\right)^{2n-k}=\sum_{k=0}^{2n}\binom{2n}{k}\left(x(x+2)\right)^{2n-k}$$

$$=\sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$$

Q14biii $\left(1+x^2+2x\right)^{2n}=\left((1+x)^2\right)^{2n}=(1+x)^{4n}$

.: coefficient of x^{2n} in the expansion of $\left(1+x^2+2x\right)^{2n}$ is $\binom{4n}{2n}$.

$$\left(1+x^2+2x\right)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$$

$$=\sum_{k=0}^{2n}\binom{2n}{k}\left[\binom{2n-k}{0}2^{2n-k}x^{2n-k}+\binom{2n-k}{1}2^{2n-k-1}x^{2n-k+1}+......\right.$$

$$\left.......+\binom{2n-k}{2n-k}2^0x^{4n-2k}\right]$$

$$=\sum_{k=0}^{2n}\left[\binom{2n}{k}\binom{2n-k}{0}2^{2n-k}x^{2n-k}+\binom{2n}{k}\binom{2n-k}{1}2^{2n-k-1}x^{2n-k+1}\right.$$

$$\left.+...........+\binom{2n}{k}\binom{2n-k}{2n-k}2^0x^{4n-2k}\right]$$

Collecting terms of x^{2n}:

$$\binom{2n}{0}\binom{2n-0}{0}2^{2n-0}x^{2n}+\binom{2n}{1}\binom{2n-1}{1}2^{2n-2}x^{2n}+......$$

$$......+\binom{2n}{n}\binom{2n-n}{2n-n}2^0x^{2n}$$

.: coefficient of x^{2n} in the expansion of $\left(1+x^2+2x\right)^{2n}$ is

$$2^{2n-0}\binom{2n}{0}\binom{2n-0}{0}+2^{2n-2}\binom{2n}{1}\binom{2n-1}{1}+...........+2^0\binom{2n}{n}\binom{2n-n}{2n-n}$$

$$=\sum_{k=0}^{n}2^{2n-2k}\binom{2n}{k}\binom{2n-k}{k}$$

$$.:\ \binom{4n}{2n}=\sum_{k=0}^{n}2^{2n-2k}\binom{2n}{k}\binom{2n-k}{k}$$

Q14ci Given $t_0=0.5$ is an approximate root of $e^t-\frac{1}{t}=0$.

Let $f(t)=e^t-\frac{1}{t}$, $f'(t)=e^t+\frac{1}{t^2}$

$$t_1=t_0-\frac{f(t_0)}{f'(t_0)}=0.5-\frac{e^{0.5}-\frac{1}{0.5}}{e^{0.5}+\frac{1}{0.5^2}}\approx 0.56$$

Q14cii Let $t=rx$, .: the two curves are $y=e^t$ and $y=\log_e\left(\frac{t}{r}\right)$,

and the derivatives are $\frac{dy}{dt}=e^t$ and $\frac{dy}{dt}=\frac{1}{t}$ respectively.

At their point of intersection, the two curves have a common tangent.

.: $e^t=\frac{1}{t}$, .: $t\approx 0.56$ from Q14cii.

Also, $\log_e\left(\frac{t}{r}\right)=e^t$.: $\log_e\left(\frac{0.56}{r}\right)\approx e^{0.56}$, $\frac{0.56}{r}\approx e^{e^{0.56}}$, .: $r\approx 0.097$

Please inform mathline@itute.com re conceptual and/or mathematical errors.