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2013 VCAA Math Methods CAS Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
C	A	E	E	A	B	A	C	E	C	B

12	13	14	15	16	17	18	19	20	21	22
C	C	D	C	E	D	B	D	A	D	D

Q1 $Period=\frac{\pi}{2\pi}=\frac{1}{2}$ C

Q2 Midpoint is $\left(\frac{d+1}{2},\frac{2+(-5)}{2}\right)=\left(\frac{d+1}{2},-\frac{3}{2}\right)$. A

Q3 $7(-a)^3+9(-a)^2-5a(-a)=0$, $a\neq 0$

$-7a^3+14a^2=0$, $-7a^2(a-2)=0$, .: $a=2$ E

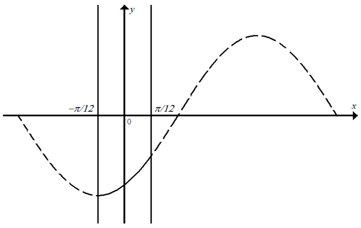
Q4 For the inverse f^{-1}, the asymptote is $x=3$ and the x-intercept is $(2,0)$. E

Q5 The maximal domain of $f+g$ is the intersection of the domains of *f* and *g*, i.e. $[-1,1)$. A

Q6 *Average rate of change over* $\left[\frac{1}{4},5\right]$ *of* $f(x)=\frac{f(5)-f(\frac{1}{4})}{5-\frac{1}{4}}$

$=\frac{\sin(10\pi)+10-(\sin\frac{\pi}{2}+\frac{1}{2})}{\frac{19}{4}}=\frac{34}{19}$ B

Q7 g must be a one to one function for g^{-1} to exist, .: $a=\frac{\pi}{12}$ A



Q8 $\begin{array}{ccc} B & D & \\ \frac{7}{10} & \frac{2}{5} & B \\ \frac{3}{10} & \frac{3}{5} & D \end{array}$ *Long term* $\Pr(B)=\frac{\frac{2}{5}}{\frac{2}{5}+\frac{3}{10}}=\frac{4}{7}$ C

Q9 $\Pr(X=15\mid X\geq 12)=\frac{\Pr(X=15)}{\Pr(X\geq 12)}$

$=\frac{binompdf(20,0.7,15)}{binomcdf(20,0.7,12,20)}\approx 0.2017$ E

Q10 $\Pr(A)=\Pr(A\cap B)+\Pr(A\cap B')=p+\frac{3p}{5}=\frac{8p}{5}$

$\Pr(B)=\Pr(B\cap A)+\Pr(B\cap A')=p+p-\frac{1}{8}=2p-\frac{1}{8}$, .: $p\neq 0$

If *A* and *B* are independent, then $\Pr(A)\Pr(B)=\Pr(A\cap B)$

.: $\frac{8p}{5}\left(2p-\frac{1}{8}\right)=p$, .: $p=\frac{3}{8}$ C

Q11 At $x=c$, $y=e^{ac}$ and $\frac{dy}{dx}=ae^{ax}=ae^{ac}$

Since the tangent passes through the origin,

.: gradient of the tangent $=\frac{e^{ac}}{c}=ae^{ac}$.

Since $e^{ac}>0$, .: $\frac{1}{c}=a$, $c=\frac{1}{a}$ B

Q12 $\frac{dx}{dt}=3e^{2t}$, .: $x=\int 3e^{2t}dt=\frac{3}{2}e^{2t}+c$ and $x=\frac{3}{2}$ when $t=0$

.: $c=0$ and $x=\frac{3}{2}e^{2t}$, .: $\frac{dx}{dt}=2x$

$y=4\cos x$, $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}=(-4\sin x)(2x)$

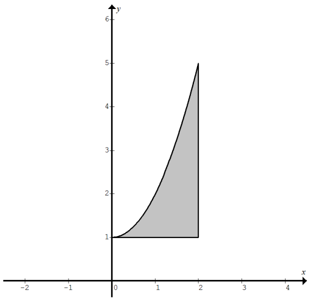
When $x=\frac{\pi}{2}$, $\frac{dy}{dt}=\left(-4\sin\frac{\pi}{2}\right)\left(2\times\frac{\pi}{2}\right)=-4\pi$ C

Q13 For $f(x)=2x$, $f(2x)=2(2x)=2f(x)$ C

Q14 $(2^1+c)1+(2^2+c)1+(2^3+c)1+(2^4+c)1=44$, $c=\frac{7}{2}$ D

Q15 Select the graph which gives the same area enclosed by the graph above or below the horizontal line $y=2$. C

Q16 When $x=5$, $y=\sqrt{5-1}=2$. The shaded region has the same area as the region bounded by $f^{-1}(x)=x^2+1$ and $x=2$.



Area of the shaded region $=\int_0^2(x^2+1-1)dx=\int_0^2(x^2)dx$ E

Q17 $\Pr(A\cap B)=\Pr(B\mid A)\Pr(A)=\Pr(A\mid B)\Pr(B)$

.: $\Pr(B\mid A)\times p^{\frac{1}{3}}=p\times p^2$, .: $\Pr(B\mid A)=p^{1+2-\frac{1}{3}}=p^{\frac{8}{3}}$ D

Q18 $g(x)=\log_2(x)$, .: $2g(8x)=2\log_2(8x)=2(\log_2(x)+\log_2 8)$

$=2\log_2(x)+6=\log_2(x^2)+6=g(x^2)+6$ B

Q19 $f(x)=e^{x\sqrt{3}}\sin x$. Let $f'(x)=\sqrt{3}e^{x\sqrt{3}}\sin x+e^{x\sqrt{3}}\cos x=0$

.: $e^{x\sqrt{3}}\left(\sqrt{3}\sin x+\cos x\right)=0$.

Since $e^{x\sqrt{3}}\neq 0$, .: $\sqrt{3}\sin x+\cos x=0$, $\tan x=-\frac{1}{\sqrt{3}}$

.: $x=\frac{5\pi}{6}$, $\frac{11\pi}{6}$ and $\frac{17\pi}{6}$, .: x-coordinate of T_3 is $\frac{17\pi}{6}$. D

Q20 Transformation T is a sequence of two single transformations. It is a reflection of function f in the x-axis, followed by a translation by 5 units in the positive x-direction. The equation of the resulting function is $y = x^2$. Find the original function f by backtracking: Translate $y = x^2$ by 5 units in the negative x-direction and then reflect in the x-axis.

$y = x^2 \to y = (x+5)^2 \to y = -(x+5)^2$ A

Q21 $f(x) = ax^3 - bx^2 + cx$, $f'(x) = 3ax^2 - 2bx + c$

f has no stationary points, .: $f'(x) \neq 0$, .: $\Delta = B^2 - 4AC < 0$

.: $4b^2 - 12ac < 0$, .: $c > \dfrac{b^2}{3a}$ D

Q22 $\Pr(X < 90) = \dfrac{150}{2000}$, $\Pr\left(Z < \dfrac{90-120}{\sigma}\right) = \dfrac{150}{2000}$

.: $\Pr\left(Z < \dfrac{-30}{\sigma}\right) = \dfrac{3}{40}$, $\dfrac{-30}{\sigma} = invNorm\left(\dfrac{3}{40}\right) = -1.4395$

.: $\sigma \approx 21$ days D

SECTION 2

Q1a $T = 25 + 2\cos\dfrac{\pi t}{8}$ and $0 \le t \le 24$

T is maximum when $\cos\dfrac{\pi t}{8} = 1$.

.: $T_{\max} = 27$ and $\dfrac{\pi t}{8} = 0$ or 2π, i.e. $t = 0$ or 16

Q1b $Period = \dfrac{2\pi}{\frac{\pi}{8}} = 16$ hours

Q1c $T = 25 + 2\cos\dfrac{\pi t}{8} = 26$, $\cos\dfrac{\pi t}{8} = \dfrac{1}{2}$, $t = \dfrac{8}{3}, \dfrac{40}{3}, \dfrac{56}{3}$

Q1d $Duration = \dfrac{8}{3} + \left(\dfrac{56}{3} - \dfrac{40}{3}\right) = 8$ hours

Q1ei $y = \sin x$, $0 \le x \le 2\pi$

When $x = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \cos x = \cos\dfrac{2\pi}{3} = -\dfrac{1}{2}$

Q1eii Gradient of $PC = \dfrac{dy}{dx}$ at $x = \dfrac{2\pi}{3}$

.: $\dfrac{0 - \frac{\sqrt{3}}{2}}{c - \frac{2\pi}{3}} = -\dfrac{1}{2}$, $c = \sqrt{3} + \dfrac{2\pi}{3}$

Q1fi $X'P' = k.XP$, $10 = k\left(\dfrac{\sqrt{3}}{2} - 0\right)$, $k = \dfrac{20}{\sqrt{3}} = \dfrac{20\sqrt{3}}{3}$

$X'C' = m.XC$, $30 = m\left(\sqrt{3} + \dfrac{2\pi}{3} - \dfrac{2\pi}{3}\right)$, $m = 10\sqrt{3}$

Q1fii P' is $\left(m\dfrac{2\pi}{3}, k\dfrac{\sqrt{3}}{2}\right) = \left(\dfrac{20\sqrt{3}\pi}{3}, 10\right)$.

Q2ai $\Pr(X \ge 10) = binomcdf\left(20, \tfrac{5}{8}, 10, 20\right) \approx 0.9153$

Q2aii $\Pr(X > 15 \mid X \ge 10) = \dfrac{\Pr(X \ge 16)}{\Pr(X \ge 10)} \approx \dfrac{0.0790}{0.9153} \approx 0.086$

Q2b $\Pr(NFFN) + \Pr(NFNF) + \Pr(NNFF)$

$= 1 \times \dfrac{1}{2} \times \dfrac{3}{4} \times \dfrac{1}{4} + 1 \times \dfrac{1}{2} \times \dfrac{1}{4} \times \dfrac{1}{2} + 1 \times \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{3}{4} = \dfrac{11}{32}$

Q2ci $E(X) = \displaystyle\int_1^3 x\left(\frac{(x-3)^3 + 64}{256}\right)dx + \int_3^5 x\left(\frac{x+29}{128}\right)dx$

$\approx 0.978125 + 2.067708 \approx 3.0458$

Q2cii $\Pr(X > 4) = \displaystyle\int_4^5 \frac{x+29}{128}dx \approx 0.26172$

Number of members taking more than 4 minutes:
$200 \times 0.26172 \approx 52$

Q3a $y = f(x) = \dfrac{3x^3}{64} - \dfrac{7x^2}{32} + \dfrac{1}{2} = \dfrac{1}{64}\left(3x^3 - 14x^2 + 32\right)$

To find G, let $f'(x) = \dfrac{1}{64}\left(9x^2 - 28x\right) = \dfrac{1}{64}x(9x - 28) = 0$ and $x > 0$. .: $x = \dfrac{28}{9}$ and $y = -\dfrac{50}{243}$, .: G is $\left(\dfrac{28}{9}, -\dfrac{50}{243}\right)$.

Q3b $\dfrac{x}{4} + \dfrac{y}{\frac{1}{2}} = 1$, $y = -\dfrac{x}{8} + \dfrac{1}{2}$

Q3ci Let $\dfrac{3x^3}{64} - \dfrac{7x^2}{32} + \dfrac{1}{2} = -\dfrac{x}{8} + \dfrac{1}{2}$, .: $\dfrac{3x^3}{64} - \dfrac{7x^2}{32} + \dfrac{x}{8} = 0$

.: $\dfrac{x}{64}\left(3x^2 - 14x + 8\right) = \dfrac{x}{64}(3x - 2)(x - 4) = 0$ and $0 < x < 2$.: $x = \dfrac{2}{3}$

Q3cii y-coordinate of D: $y = -\dfrac{x}{8} + \dfrac{1}{2} = -\dfrac{1}{8}\left(\dfrac{2}{3}\right) + \dfrac{1}{2} = \dfrac{5}{12}$

$AD = \sqrt{\left(\dfrac{2}{3} - 0\right)^2 + \left(\dfrac{5}{12} - \dfrac{1}{2}\right)^2} = \dfrac{\sqrt{65}}{12}$ km

Q3di Let H km be the length of the column.

$H = \left(-\dfrac{x}{8} + \dfrac{1}{2}\right) - \left(\dfrac{3x^3}{64} - \dfrac{7x^2}{32} + \dfrac{1}{2}\right)$

$= -\dfrac{1}{64}\left(3x^3 - 14x^2 + 8x\right)$ and $\dfrac{2}{3} \le x \le 4$

Let $\dfrac{dH}{dx} = -\dfrac{1}{64}\left(9x^2 - 28x + 8\right) = 0$, .: $x = \dfrac{14 + 2\sqrt{31}}{9}$ (≈ 2.7928)

Q3dii $H_{\max} \approx -\dfrac{1}{64}\left(3 \times 2.7928^3 - 14 \times 2.7928^2 + 8 \times 2.7928\right)$

≈ 0.336 km, i.e. 336 m

Q3e $V(x) = k\sqrt{x} - mx^2$ and $0 \le x \le 4$

$B(4,0) \to 2k - 16m = 0$, .: $k = 8m$

Q3f .: $V(x)=k\sqrt{x}-mx^2=8m\sqrt{x}-mx^2=m\left(8\sqrt{x}-x^2\right)$

Let $\frac{dV}{dx}=m\left(\frac{4}{\sqrt{x}}-2x\right)=0$, .: $\frac{4}{\sqrt{x}}-2x=0$, $x=2^{\frac{2}{3}}$

Q3g When $m=10$, $V(x)=10\left(8\sqrt{x}-x^2\right)$

$V_{\max}=10\left(8\sqrt{2^{\frac{2}{3}}}-\left(2^{\frac{2}{3}}\right)^2\right)\approx 75.6$ km h^{-1}

Q3h $V(x)=m\left(8\sqrt{x}-x^2\right)$, $V_{\max}=m\left(8\sqrt{2^{\frac{2}{3}}}-\left(2^{\frac{2}{3}}\right)^2\right)=120$

.: $m=10\times 2^{\frac{2}{3}}$

Q4ai $y=g(x)=\frac{16-x^2}{4}$, $g'(x)=-\frac{x}{2}$

Point *C*: $x=0$, $y=4$, $(0,4)$; point *B*: $y=0$, $x=4$, $(4,0)$

Gradient of BC $=\frac{4-0}{0-4}=-1$, .: $-\frac{x}{2}=-1$, $x=2$ and .: $y=3$

.: point *A* is $(2,3)$

Equation of the tangent at *A*: $y-3=-1(x-2)$, $y=-x+5$

Q4aii The tangent cuts the axes at $(5,0)$ and $(0,5)$.

Area of the shaded region $=\frac{1}{2}(5)(5)-\int_0^4\frac{16-x^2}{4}\,dx=\frac{25}{2}-\frac{32}{3}=\frac{11}{6}$

Q4b Let Q be $(x,g(x))$, i.e. $\left(x,\frac{16-x^2}{4}\right)$, $x>0$

Let L be the distance OQ, $L=\sqrt{x^2+\left(\frac{16-x^2}{4}\right)^2}$.

$\frac{dL}{dx}=0$ at $x=2\sqrt{2}$, $L_{\min}=2\sqrt{3}$

Q4c $y=g(x)=\frac{16-x^2}{4}$, $g'(x)=-\frac{x}{2}$

Point *P* is $\left(x,\frac{16-x^2}{4}\right)$ and point *D* is $(0,k)$.

Gradient of PD $=g'(x)$, .: $\frac{\frac{16-x^2}{4}-k}{0-x}=-\frac{x}{2}$

Solve for *x* in terms of *k*: $x=2\sqrt{k-4}$

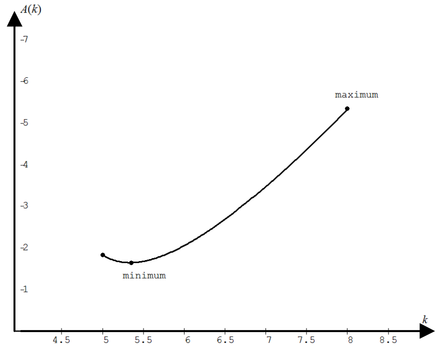
.: *gradient of the tangent* $=g'\left(2\sqrt{k-4}\right)=-\sqrt{k-4}$, $5\le k\le 8$

Q4di Equation of the tangent at *P*: $y=\left(-\sqrt{k-4}\right)x+k$

.: the tangent cuts the axes at $\left(\frac{k}{\sqrt{k-4}},0\right)$ and $(0,k)$.

$A(k)=\frac{1}{2}\left(\frac{k}{\sqrt{k-4}}\right)(k)-\frac{32}{3}=\frac{k^2}{2\sqrt{k-4}}-\frac{32}{3}$

Q4dii Sketch $A(k)=\frac{k^2}{2\sqrt{k-4}}-\frac{32}{3}$, $5\le k\le 8$.



$A_{\max}=\frac{8^2}{2\sqrt{8-4}}-\frac{32}{3}=\frac{16}{3}$ when $k=8$

Q4diii Let $\frac{dA}{dk}=0$ and solve for $5\le k\le 8$, $k=\frac{16}{3}$

$A_{\min}=\frac{\left(\frac{16}{3}\right)^2}{2\sqrt{\frac{16}{3}-4}}-\frac{32}{3}=\frac{64\sqrt{3}-96}{9}$ when $k=\frac{16}{3}$

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