2015 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	E	C	E	C	B	B	A	D	E	B

12	13	14	15	16	17	18	19	20	21	22
E	C	E	E	D	D	A	E	C	D	E

Q1 $x = f(y)$ is the inverse of $y = f(x)$. If they do intersect, the intersecting point is at the line $y = x$. D

Q2 $2^x - 2^{\frac{x+b}{2}} + \frac{1}{2} = 0$, $\left(2^{\frac{x}{2}}\right)^2 - 2^{\frac{b}{2}}\left(2^{\frac{x}{2}}\right) + \frac{1}{2} = 0$

$\Delta = \left(-2^{\frac{b}{2}}\right)^2 - 4(1)\left(\frac{1}{2}\right) = 2^b - 2 > 0$ for two distinct solutions

.: $b > 1$ E

Q3 $g(x) < 0$ for $x \in (-2a, -a)$ and $g(-a)$ is undefined,
.: $f(g(x))$ is defined for $x \in R \setminus (-2a, -a]$ C

Q4 Before the transformations: gradient $= \frac{\text{rise}}{\text{run}} = \frac{m}{1}$

After the transformations: gradient $= \frac{2 \times m}{0.5 \times 1} = 4m$ E

Q5 $y = \log_e |2x|$, $\frac{dy}{dx} = \frac{1}{x} = \frac{1}{a}$ at $x = a$

.: the gradient of the perpendicular at $x = a$ is $\frac{-1}{\frac{1}{a}} = -a$ C

Q6 Estimate the signed area of the regions bounded by the curve and the x-axis, then divide by 20. B

Q7 For a concave graph, points of inflection exist in pairs. B

Q8 $f(x) = \left(\frac{\sin(nx)}{2}\right)^2 = \frac{1}{8} - \frac{1}{8}\cos(2nx)$, $T = \frac{2\pi}{2n} = \frac{\pi}{n}$ A

Q9 There are $2n - 1$ asymptotes inside interval $(0, 2\pi)$ D

Q10 $f(-x) + f(x) = 0$, .: $f(x)$ is an odd function. E

Q11 $\int_1^0 f(x - 0.5)dx = 2$, .: $\int_0^1 f(x - 0.5)dx = -2$

$f(1 - 2x)$ is the reflection (in the y-axis) and the dilation (by a factor of 0.5 in the x-direction) of $f(x - 0.5)$,

.: $\int_{-0.5}^0 f(1 - 2x)dx = -1$ B

Q12 The local minimum of $f(x)$ is at $x > 0$. For the graph of $y = f(x)$ to cross the negative x-axis only once, the y-intercept must be at or above the origin, i.e. $f(0) = b - a \geq 0$, .: $b \geq a$.
Note: $a > \sqrt{1 + 3a} - 1$ for $a > 1$, .: D is not the answer. E

Q13 $1 - 2^{ax+1} - 2 \times 2^{2ax} + 4 \times 2^{3ax} = 0$
$(1 - 2 \times 2^{ax}) - 2 \times 2^{2ax}(1 - 2 \times 2^{ax}) = 0$
$(1 - 2 \times 2^{ax})(1 - 2 \times 2^{2ax}) = 0$, $(1 - 2^{ax+1})(1 - 2^{2ax+1}) = 0$
.: $2^{ax+1} = 1$ or $2^{2ax+1} = 1$
.: $x = -\frac{1}{a}$ or $x = -\frac{1}{2a}$ C

Q14 $\log_e\left(\frac{x}{b}\right) > \log_e(x + a - ab)$, .: $\frac{x}{b} > x + a - ab$ where $x > 0$, $x + a - ab > 0$ and $b > a > 1$, .: $x > ab - a$
Also, from $\frac{x}{b} > x + a - ab$, $x < ab$
.: $(ab - a) < x < ab$ E

Q15 $y = a\log_e(bx)$ and $y = \frac{1}{b}e^{\frac{x}{a}}$ are inverse of each other.
Their common point is on the line $y = x$.
.: $\frac{dy}{dx} = \frac{a}{x} = 1$ and $\frac{dy}{dx} = \frac{1}{ab}e^{\frac{x}{a}} = 1$ at the common point
.: $x = a$ and .: $ab = e$ E

Q16 $2\sin\left(\frac{3\pi}{2} - kx\right) + \sqrt{3} = 0$, .: $-2\cos(kx) + \sqrt{3} = 0$

$\cos(kx) = \frac{\sqrt{3}}{2}$, $kx = 2n\pi \pm \frac{\pi}{6} = \left(\frac{12n \pm 1}{6}\right)\pi$

.: $x = \left(\frac{12n \pm 1}{6k}\right)\pi$ D

Q17 $\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} a \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 - a \\ a \end{bmatrix}$ D

Q18 $a\sin 0 + a\sin\frac{\pi}{4} + a\sin\frac{\pi}{2} + a\sin\frac{3\pi}{4} + a\sin\pi = 1$

.: $a = \frac{1}{1 + \sqrt{2}}$

.: $\Pr\left(X = \frac{\pi}{2}\right) = a\sin\frac{\pi}{4} = \frac{1}{1 + \sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2 + \sqrt{2}}$

$= \frac{2 - \sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}}$ A

Q19 It is a binomial distribution of X with parameters $n = 5$ and $p = 0.9$.
$\mu = np = 4.5$ and $\sigma = \sqrt{np(1 - p)} \approx 0.67$ E

Q20 $\mu - 2\sigma = 1$ and $\mu + \sigma = 4$, .: $\mu = 3$ and $\sigma = 1$

$\Pr\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{3\sigma}{2}\right) = \Pr(2.5 < X < 4.5) \approx 0.6241$ C

Q21 $(3.5-2.5)p + (4.3-3.8)3p = 1$, .: $p = 0.4$

.: $(3.5-2.5)p = 0.4 < 0.5$

.: the median m is in $3.8 \le x \le 4.3$

.: $(4.3-m)3 \times 0.4 = 0.5$, $m \approx 3.88$ D

Q22

	A	A'	
B	x	$0.7-x$	0.7
B'	$0.5-x$	$x-0.2$	0.3
	0.5	0.5	1

where $x \ge 0$, $0.7-x \ge 0$, $0.5-x \ge 0$ and $x-0.2 \ge 0$

.: $0.2 \le x \le 0.5$ E

SECTION 2

Q1a i $f(x) - g(x) = (x-a)\left(x-\frac{5}{2}\right) - \left(x^2 - 5x + \frac{25}{4}\right)$

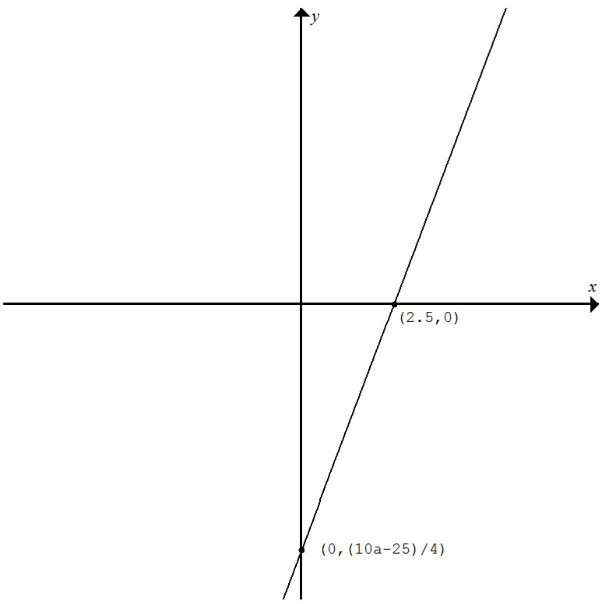
$= (x-a)\left(x-\frac{5}{2}\right) - \left(x-\frac{5}{2}\right)^2 = \left(\frac{5}{2}-a\right)\left(x-\frac{5}{2}\right)$

$= \left(\frac{5}{2}-a\right)x + \left(\frac{10a-25}{4}\right)$.: $B = \frac{5}{2} - a$ and $C = \frac{10a-25}{4}$

Q1a ii $y = \left(\frac{5}{2}-a\right)x + \left(\frac{10a-25}{4}\right)$

x-intercept: Let $y = 0$, $x = \frac{5}{2}$

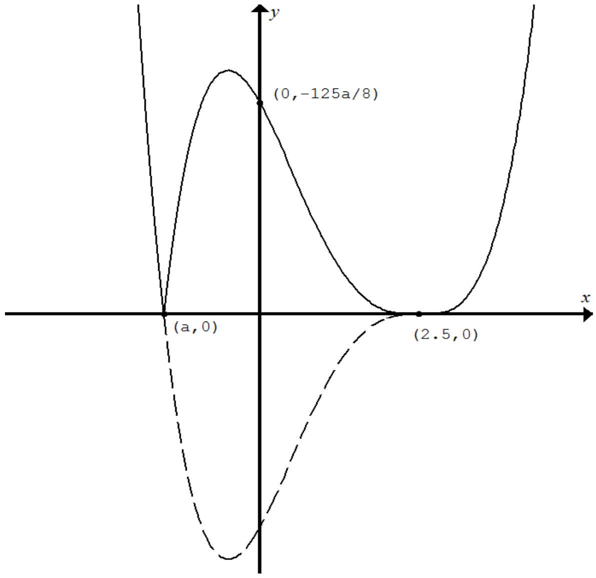
y-intercept: $y = C = \frac{10a-25}{4}$



Q1b $y = (x-a)\left(x-\frac{5}{2}\right) + \left(x-\frac{5}{2}\right)^2 = 2\left(x - \frac{5}{4} - \frac{a}{2}\right)\left(x-\frac{5}{2}\right)$

At the turning point, $x = \frac{1}{2}\left(\frac{5}{4} + \frac{a}{2} + \frac{5}{2}\right) = \frac{1}{2}\left(\frac{15}{4} + \frac{a}{2}\right)$

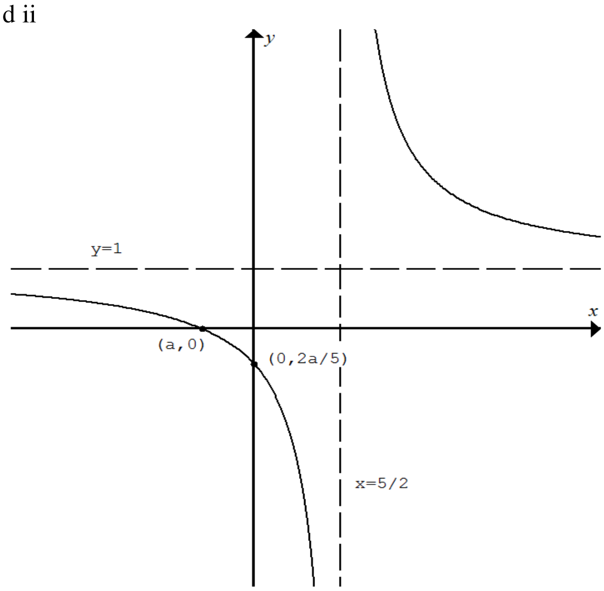
Q1c $y = |f(x)||g(x)| = |f(x)g(x)| = \left|(x-a)\left(x-\frac{5}{2}\right)^3\right|$



Q1d i $y = \frac{f(x)}{g(x)} = \frac{(x-a)\left(x-\frac{5}{2}\right)}{\left(x-\frac{5}{2}\right)^2} = \frac{x-a}{x-\frac{5}{2}}$

$= \frac{x - \frac{5}{2} + \frac{5}{2} - a}{x - \frac{5}{2}} = 1 + \frac{\frac{5}{2} - a}{x - \frac{5}{2}}$, .: $A = \frac{5}{2} - a$

Q1d ii

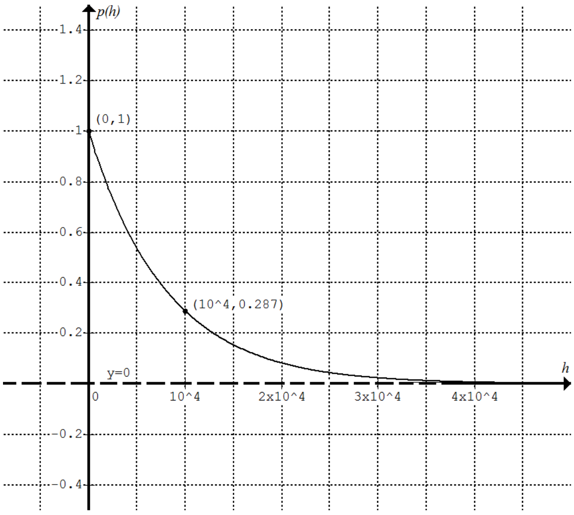


Q2a $p(h)=p_0\times10^{kh}$, $p(0)=p_0\times10^0=1.00$, .: $p_0=1.00$

$p(1.00\times10^4)=1.00\times10^{(1.00\times10^4)k}=0.287$

.: $10^4k=\log_{10}0.287$, .: $k=-5.42\times10^{-5}$

Q2b



Q2c $p=p_0\times10^{kh}$, $10^{kh}=\dfrac{p}{p_0}$,

$h=\dfrac{1}{k}\log_{10}\dfrac{p}{p_0}\approx1.84\times10^4\log_{10}p$

Q2d $\dfrac{p_{\text{cabin}}}{p_0}=\dfrac{p(1600)}{p_0}=10^{-5.42\times10^{-5}\times1600}\approx0.82$

Q2e Average rate of change $=\dfrac{\Delta p}{\Delta h}=\dfrac{0.82-1.00}{1600}\approx-1.13\times10^{-4}$

Average rate of decrease $\approx1.13\times10^{-4}$ atmosphere per metre

Q2f The two graphs have the same shape, intercept and asymptote.

Q2g $p=1.00\times10^{-5.42\times10^{-5}h}=e^{ch}$

.: $e^c=10^{-5.42\times10^{-5}}$, $c=\log_e\left(10^{-5.42\times10^{-5}}\right)=-1.25\times10^{-4}$

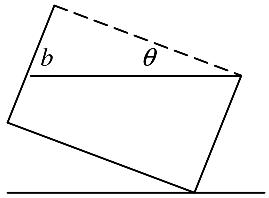
Q2h $\dfrac{dp}{dh}=-1.25\times10^{-4}\times e^{-1.25\times10^{-4}\times1600}\approx-1.02\times10^{-4}$

Rate of decrease $=1.02\times10^{-4}$ atmosphere per metre

Q2i $p=e^{ch}$, $\dfrac{dp}{dh}=ce^{ch}=cp$, .: $\dfrac{dp}{dh}\propto p$ and the constant of proportionality $c=-1.25\times10^{-4}$.

Q3a $\tan\alpha°=\dfrac{\sqrt{3}}{3}$, $\alpha=30$

Q3b



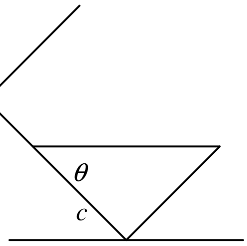
$b=3\tan\theta°$

Volume of water spilled out $=\dfrac{1}{2}\times3\times3\tan\theta°\times3$

Volume of water remains $=3\times3\times\sqrt{3}-\dfrac{1}{2}\times3\times3\tan\theta°\times3$

$=\dfrac{27}{2}\left(\dfrac{2}{\sqrt{3}}-\tan\theta°\right)$

Q3c

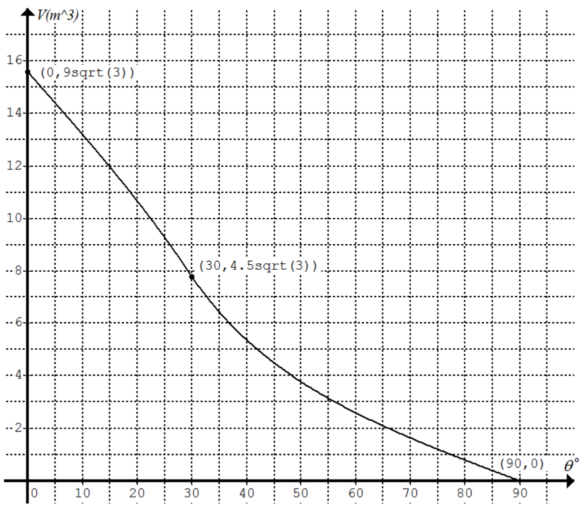


$c=\dfrac{\sqrt{3}}{\tan\theta°}$

Volume of water remains

$=f(\theta)=\dfrac{1}{2}\times\dfrac{\sqrt{3}}{\tan\theta°}\times\sqrt{3}\times3=\dfrac{9}{2\tan\theta°}$

Q3d



Q3e $V=\frac{27}{2}\left(\frac{2}{\sqrt{3}}-\tan\theta°\right)=\frac{27}{2}\left(\frac{2}{\sqrt{3}}-\tan\frac{\pi\theta}{180}\right)$

$\frac{dV}{d\theta}=-\frac{27}{2}\times\frac{\pi}{180}\sec^2\frac{\pi\theta}{180}=-\frac{3\pi}{40}\sec^2\theta°=-\frac{3\pi}{40\cos^2\theta°}$

$V=\frac{9}{2\tan\theta°}=\frac{9}{2}\left(\tan\frac{\pi\theta}{180}\right)^{-1}$

$\frac{dV}{d\theta}=-\frac{\pi}{180}\times\frac{9}{2\tan^2\frac{\pi\theta}{180}}\times\sec^2\frac{\pi\theta}{180}=-\frac{\pi\sec^2\theta°}{40\tan^2\theta°}=-\frac{\pi}{40\sin^2\theta°}$

Q3f When $\theta\to 30$, $-\frac{3\pi}{40\cos^2\theta°}=-\frac{\pi}{10}$, $-\frac{\pi}{40\sin^2\theta°}=-\frac{\pi}{10}$

.: both derivatives approach the same value $-\frac{\pi}{10}$ m^3 per degree

Q3g $\frac{dV}{dt}=\frac{dV}{d\theta}\times\frac{d\theta}{dt}=-\frac{\pi}{10}\times 0.2=-\frac{\pi}{50}$

Rate of decrease $=\frac{\pi}{50}$ m^3 per second

Q4a $\mu=65$, $\sigma=10$

Note: Some of the following answers may differ in the second decimal place because of using CAS ($\mu=65$, $\sigma=10$) or working from the given graph.

Q4b $\Pr(60<X<70)\approx 0.37$ from graph

Q4c $0.31=31\%$ from graph

Q4d $100\%-31\%=69\%$

Q4e $\Pr(\text{B grade})=\Pr(70<X<80)\approx 0.26$ from graph

Q4f

$\Pr(\text{at least one B grade})=1-\Pr(\text{none})\approx 1-(1-0.26)^3\approx 0.59$

Q4g $\Pr(\text{B grade or above})\approx 1-0.68=0.32$

Mean number of students $=np=3\times 0.32=0.96$

Q4h $\Pr(\text{B grade or below})\approx 0.94$

$\sigma=\sqrt{np(1-p)}=\sqrt{3\times 0.94\times 0.06}\approx 0.41$

Q4i $\Pr(\text{B grade} \mid \text{higher than D grade})=\frac{0.26}{0.68}\approx 0.382$

$\Pr(X=2)={}^3C_2\times 0.382^2\times(1-0.382)\approx 0.27$

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