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2015 NSW BOS Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	C	B	C	A	D	B	B	D	C

Q1 Remainder $= (-3)^3 - 6(-3) = -9$ — A

Q2 $\frac{dN}{dt} = k80e^{kt} = k(N-100)$ — C

Q3 $(3+x)x = (6+4)\times 4$, $x > 0$

$x^2 + 3x - 40 = 0$, $(x-5)(x+8) = 0$, .: $x = 5$ — B

Q4 — C

Q5 $y = \frac{3x}{(x+1)(x+2)} = \frac{\frac{3}{x}}{\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}$, as $x \to \pm\infty$, $y \to 0$ — A

Q6 $-1 \le 2x \le 1$, $-\frac{1}{2} \le x \le \frac{1}{2}$ — D

Q7 $\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^k = \frac{\pi}{3}$, $\sin^{-1}\left(\frac{k}{2}\right) = \frac{\pi}{3}$, $\frac{k}{2} = \frac{\sqrt{3}}{2}$, $k = \sqrt{3}$ — B

Q8 $\lim_{x\to 3} \frac{\frac{\sin(x-3)}{(x-3)}}{x+2} \to \frac{1}{5}$ — B

Q9 $a = 2\times 3 = 6$; $\frac{2\pi}{n} = \frac{1}{2}\times\frac{2\pi}{4}$, .: $n = 8$ — D

Q10 $\cos\left(2t - \frac{\pi}{3}\right) = 0$, $2t - \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$, $2t = \frac{5\pi}{6}, \frac{11\pi}{6}$

.: $t = \frac{5\pi}{12}, \frac{11\pi}{12}$, .: $P\left(\frac{11\pi}{12}, 0\right)$ — C

Section II

Q11a $\int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Q11b $\tan\theta = \frac{(-3)-(2)}{1+(-3)(2)} = \frac{-5}{1-6} = 1$, $\theta = \frac{\pi}{4}$ $(45°)$

Q11c $\frac{4}{x+3} \ge 1$, $\frac{4}{x+3} - 1 \ge 0$ where $x+3 > 0$, .: $x > -3$

For $x+3>0$, $4 \ge x+3$, $x \le 1$.: $-3 < x \le 1$

Q11d

$A\cos(x+\alpha) = A\cos x\cos\alpha - A\sin x \sin\alpha = 5\cos x - 12\sin x$

.: $A\cos\alpha = 5$ and $A\sin\alpha = 12$.: $A = 13$ and $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

.: $5\cos x - 12\sin x = 13\cos\left(x + \tan^{-1}\left(\frac{12}{5}\right)\right)$

Q11e Let $u = 2x-1$, $x = \frac{u+1}{2}$, $dx = \frac{1}{2}du$

When $x = 1$, $u = 1$; when $x = 2$, $u = 3$

$\int_1^2 \frac{x}{(2x-1)^2}dx = \int_1^3 \frac{u+1}{4u^2}du = \frac{1}{4}\int_1^3\left(\frac{1}{u} + \frac{1}{u^2}\right)du$

$= \frac{1}{4}\left[\ln u - \frac{1}{u}\right]_1^3 = \frac{1}{4}\left(\ln 3 + \frac{2}{3}\right)$

Q11fi $P(x) = x^3 - kx^2 + 5x + 12$

$P(3) = 3^3 - k3^2 + 5\times 3 + 12 = 0$, .: $k = 6$

Q11fii $P(x) = x^3 - 6x^2 + 5x + 12$

$= (x-3)(x^2 - 3x - 4) = (x-3)(x-4)(x+1)$

Hence the zeros of $P(x)$ are: $^{-}1, 3$ and 4

Q12ai $\angle ACB = 90° - 30° = 60°$

Q12aii $\angle ADX = \angle ACD = 30°$

Q12aiii $\angle CAB = \angle CDB$

(angles on the circumference subtended by the same arc *CB*)

$\angle CDB = 100° - 30° = 70°$

(interior and exterior angles of a triangle)

.: $\angle CAB = 70°$

Q12bi $(p+q)x - 2y - 2apq = 0$ passes through the focus $(0, a)$

.: $(p+q)0 - 2a - 2apq = 0$, .: $pq = -1$

Q12bii $2ap = 8a$, .: $p = 4$ and .: $q = \frac{-1}{4}$.: $Q\left(\frac{-a}{2}, \frac{a}{16}\right)$

Q12ci ΔOAM : $\frac{OA}{OM} = \cot 15°$, $OA = OM\cot 15° = h\cot 15°$

Q12cii ΔOBM : $\frac{OB}{OM} = \cot 13°$, $OB = OM\cot 13° = h\cot 13°$

ΔOAB : $OB^2 - OA^2 = AB^2$, $h^2(\cot^2 13° - \cot^2 15°) = 2000^2$

$h = \frac{2000}{\sqrt{\frac{1}{\tan^2 13°} - \frac{1}{\tan^2 15°}}} \approx 910$

Q12di The cosine rule: $160^2 = r^2 + r^2 - 2r^2\cos\theta$

.: $160^2 = 2r^2(1-\cos\theta)$

Q12dii $r\theta = 200$, $r = \frac{200}{\theta}$

$160^2 = 2r^2(1-\cos\theta)$, $160^2 = 2\left(\frac{200}{\theta}\right)^2(1-\cos\theta)$

$160^2\theta^2 = 2(200)^2(1-\cos\theta)$, .: $8\theta^2 + 25\cos\theta - 25 = 0$

Q12diii Let $f(\theta) = 8\theta^2 + 25\cos\theta - 25$, $f'(\theta) = 16\theta - 25\sin\theta$

$\theta_1 = \pi$, $\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)} = \pi - \frac{8\pi^2 + 25\cos\pi - 25}{16\pi - 25\sin\pi} \approx 2.57$

Q13ai The particle is at rest ($v=0$) for $x=3$, 7.

Q13aii Maximum speed $=\sqrt{11}$ m s^{-1}

Q13aiii The amplitude of the oscillation is $a=2$.
Maximum speed occurs at the centre of the oscillation, .: $c=5$

.: $v^2=n^2\left(4-(x-5)^2\right)$.: $11=4n^2$, $n=\dfrac{\sqrt{11}}{2}$

Q13bi The third term is $\binom{18}{2}(2x)^{16}\left(\dfrac{1}{3x}\right)^2=\binom{18}{2}\dfrac{2^{16}}{3^2}x^{14}$

.: $a_2=\binom{18}{2}\dfrac{2^{16}}{3^2}$

Q13bii The term independent of x is

$$\binom{18}{9}(2x)^9\left(\frac{1}{3x}\right)^9=\binom{18}{9}\frac{2^9}{3^9}=\binom{18}{9}\left(\frac{2}{3}\right)^9$$

Q13c For $n=1$, $\dfrac{1}{2!}=1-\dfrac{1}{2!}$ is true

Assume it is true for $n=k$,

i.e. $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{k}{(k+1)!}=1-\dfrac{1}{(k+1)!}$

For $n=k+1$,

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$$

$$=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{k+2}{(k+2)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}$$

.: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$ for all integers $n\ge 1$.

Q13di $f(x)=\cos^{-1}(x)+\cos^{-1}(-x)$, $-1\le x\le 1$

$f'(x)=\dfrac{-1}{\sqrt{1-x^2}}+\dfrac{(-1)(-1)}{\sqrt{1-x^2}}=0$, .: $f(x)$ is constant

Q13dii Let $x=1$, $\cos^{-1}(1)+\cos^{-1}(-1)=0+\pi=\pi$

.: $f(x)=\cos^{-1}(x)+\cos^{-1}(-x)=\pi$, .: $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$

Q14ai Given $x=Vt\cos\theta$, $y=Vt\sin\theta-\dfrac{1}{2}gt^2$

Let $y=0$, $Vt\sin\theta-\dfrac{1}{2}gt^2=t\left(V\sin\theta-\dfrac{1}{2}gt\right)=0$

.: $t=0$ or $t=\dfrac{2V\sin\theta}{g}$

.: $x=0$ or $x=\dfrac{V^2 2\sin\theta\cos\theta}{g}=\dfrac{V^2\sin 2\theta}{g}$ respectively

.: the range is $\dfrac{V^2\sin 2\theta}{g}$

Q14aii When $t=\dfrac{2V}{\sqrt{3}g}$ and $\theta=\dfrac{\pi}{3}$

$\dfrac{dx}{dt}=V\cos\dfrac{\pi}{3}=\dfrac{V}{2}$ and $\dfrac{dy}{dt}=V\sin\dfrac{\pi}{3}-\dfrac{2V}{\sqrt{3}}=\dfrac{\sqrt{3}V}{2}-\dfrac{2V}{\sqrt{3}}=\dfrac{-V}{2\sqrt{3}}$

.: $\tan\phi=\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\dfrac{1}{\sqrt{3}}$, .: $\phi=-\dfrac{\pi}{6}$

The projectile travels at angle $\dfrac{\pi}{6}$ with (below) the horizontal.

Q14aiii The negative sign in $\phi=-\dfrac{\pi}{6}$ indicates the particle is travelling downwards.

Q14bi $\ddot{x}=x-1$, $\dfrac{d}{dx}\left(\dfrac{\dot{x}^2}{2}\right)=x-1$,

$\dfrac{\dot{x}^2}{2}=\displaystyle\int(x-1)dx=\dfrac{(x-1)^2}{2}+c_1$

To satisfy the initial conditions, $\dot{x}=1$ when $x=0$, $c_1=0$ and $\dot{x}^2=(x-1)^2$, .: $\dot{x}=-(x-1)$, i.e. $\dot{x}=1-x$

Q14bii $\dot{x}=1-x$, $\dfrac{dx}{dt}=1-x$, $\dfrac{dt}{dx}=\dfrac{1}{1-x}$, $t=\displaystyle\int\dfrac{1}{1-x}dx$,

$t=-\ln|1-x|+c_2$

$x=0$ when $t=0$, .: $c_2=0$ and $t=-\ln|1-x|$, .: $1-x=e^{-t}$

.: $x=1-e^{-t}$

Q14biii $\lim\limits_{t\to\infty}x=1$

Q14ci $\Pr(A\text{ wins the seventh game and any 4 of the first 6 games})$

$=\binom{6}{4}\left(\dfrac{1}{2}\right)^7$

Q14cii $\binom{4}{4}\left(\dfrac{1}{2}\right)^5+\binom{5}{4}\left(\dfrac{1}{2}\right)^6+\binom{6}{4}\left(\dfrac{1}{2}\right)^7$

Q14ciii Player *A* must win a total of $(n+1)$ games in at most $(2n+1)$ games.

If *A* did not win the $(2n+1)$th game, *B* gets the prize.

$\Pr(A\text{ gets the prize})=\dfrac{1}{2}$

$$\binom{n}{n}\left(\frac{1}{2}\right)^{n+1}+\binom{n+1}{n}\left(\frac{1}{2}\right)^{n+2}+\binom{n+2}{n}\left(\frac{1}{2}\right)^{n+3}+\cdots+\binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1}=\frac{1}{2}$$

Multiply both sides by 2^{2n+1}:

$$\binom{n}{n}2^n+\binom{n+1}{n}2^{n-1}+\binom{n+2}{n}2^{n-2}+\cdots+\binom{2n}{n}=2^{2n}$$

Please inform mathline@itute.com re conceptual and/or mathematical errors.