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# 2016 NSW BOS Mathematics Extension 1 Solutions



## Section I

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| D | B | A | C | A | D | C | B | A | A |

Q1 First term $= 2\times 1+1=3$, last term $= 2\times 20+1=41$ **D**

Q2 Remainder $= 2(2)^3-10(2)^2+6(2)+2=-10$ **B**

Q3 $\dfrac{\tan 2x-\tan x}{1+\tan 2x\tan x}=\tan(2x-x)=\tan x$ **A**

Q4 $360-(180-40)=220$ **C**

Q5 $\int \sin^2 2x\,dx=\int \frac{1}{2}(1-\cos 4x)dx=\frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+c$ **A**

Q6 $(2\sin x-1)(\sin x-3)=0$, $\sin x=\frac{1}{2}$, $x=\frac{\pi}{6},\frac{5\pi}{6},\cdots$ **D**

Q7 $\dfrac{dx}{dt}=20\sin 4t-48\cos 4t$, max $v=\sqrt{20^2+48^2}=52$ **C**

Q8 ${}^{18}C_{11}-{}^{15}C_{11}=30459$ **B**

Q9 $f(1)<1$, gradient > 1 at $x=1$ .: $f'(1)>1$, gradient is decreasing in the vicinity of $x=1$, $f''(1)<0$. **A**

Q10 $p(x)=ax^3+bx^2+cx-6$, $a,b>0$, .: either choice A or B.
$p'(x)=3ax^2+2bx+c$, $p''(x)=6ax+2b$, let $p''(x)=0$, .: the point of inflection is at $x=-\frac{b}{3a}<0$, also $p''(0)=2b>0$, i.e. gradient is increasing in the vicinity of $x=0$. **A**

## Section II

Q11a Inverse of $y=x^3-2$ is $x=y^3-2$, .: $y=(x+2)^{\frac{1}{3}}$

Q11b $\int x\sqrt{x-4}\,dx=\int (u+4)u^{\frac{1}{2}}\,du=\frac{2}{5}u^{\frac{5}{2}}+\frac{8}{3}u^{\frac{3}{2}}+c$
$=\frac{2}{5}(x-4)^{\frac{5}{2}}+\frac{8}{3}(x-4)^{\frac{3}{2}}+c$

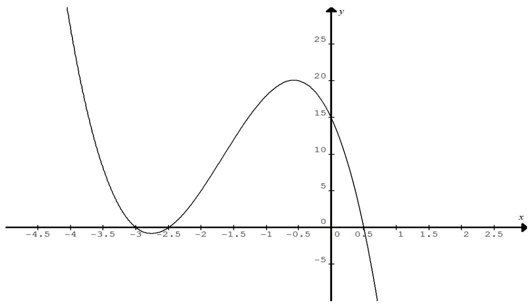
Q11c $\frac{d}{dx}\left(3\tan^{-1}2x\right)=\frac{3}{1+(2x)^2}\times 2=\frac{6}{1+4x^2}$

Q11d $\lim_{x\to 0}\frac{2\sin x\cos x}{3x}=\lim_{x\to 0}\frac{\sin 2x}{3x}=\frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{2}{3}\times 1=\frac{2}{3}$

Q11e $\frac{3}{2x+5}-x>0$, $x\neq -\frac{5}{2}$

$\frac{3(2x+5)}{(2x+5)^2}>x$, $3(2x+5)>x(2x+5)^2$, $3(2x+5)-x(2x+5)^2>0$

$(2x+5)(3-x(2x+5))>0$, $(2x+5)(3+x)(1-2x)>0$




$x<-3$, $-\frac{5}{2}<x<\frac{1}{2}$

Q11fi $\Pr(\text{exactly 1 of first 3})=3\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^2=\frac{36}{125}$

Q11fii $\Pr(\text{at least 2 of first 6})=1-\Pr(0)-\Pr(1)$

$=1-\left(\frac{2}{5}\right)^6-6\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^5=\frac{2997}{3125}$

Q12ai Similar triangles, $\frac{r}{h}=\frac{5}{20}$, $r=\frac{h}{4}$

Q12aii $v=\frac{1}{3}\pi r^2 h=\frac{\pi}{3}\left(\frac{h}{4}\right)^2 h=\frac{\pi}{48}h^3$, $\frac{dv}{dh}=\frac{\pi}{16}h^2$

Q12aiii $A=\pi r^2=\frac{\pi h^2}{16}$, $\frac{dA}{dh}=\frac{\pi h}{8}$, $\frac{dh}{dt}=\frac{\frac{dA}{dt}}{\frac{dA}{dh}}=\frac{-0.04}{\frac{\pi h}{8}}=\frac{-0.32}{\pi h}$

Q12aiv $\frac{dv}{dt}=\frac{dv}{dh}\times\frac{dh}{dt}=\frac{\pi h^2}{16}\times\frac{-0.32}{\pi h}=-0.02h$

When $h=10$, $\frac{dv}{dt}=-0.2\text{ cm}^3$ per s

Q12bi Given $\frac{dx}{dt}=ky$ where $k$ is a constant and $x+y=500$

.: $\frac{dx}{dt}=k(500-x)$. When $t=0$, $x=0$ and $\frac{dx}{dt}=2$, .: $k=0.004$

.: $\frac{dx}{dt}=0.004(500-x)$

Q12bii $x=500-Ae^{-0.004t}$

$\frac{dx}{dt}=0.004Ae^{-0.004t}=0.004(500-x)$

When $t=0$, $x=0$, $0=500-Ae^{0}$, .: $A=500$

Q12ci $y=\tan x$, $\frac{dy}{dx}=\sec^2 x=\frac{1}{\cos^2 x}=m_1$

$y=\cos x$, $\frac{dy}{dx}=-\sin x=m_2$

At $x=\alpha$, $\cos x=\tan x=\frac{\sin x}{\cos x}$, .: $\cos^2 x=\sin x$

$m_1 m_2=-\frac{\sin x}{\cos^2 x}=-\frac{\sin x}{\sin x}=-1$

.: the curves are perpendicular at $x=\alpha$

Q12cii Let $f(x)=\tan x-\cos x$, $f'(x)=\frac{1}{\cos^2 x}+\sin x$

$x_1=1$, $x_2\approx 1-\frac{f(1)}{f'(1)}=1-\frac{\tan 1-\cos 1}{\frac{1}{\cos^2 1}+\sin 1}\approx 0.76$, .: $\alpha\approx 0.76$

Q13ai Amplitude $\frac{9-1}{2}=4$, centre $=1+4=5$, period $T=\frac{25}{2}$

.: $\frac{2\pi}{n}=\frac{25}{2}$, $n=\frac{4\pi}{25}$. First high tide at $t=0$, .: use the cosine function to model the tide. Let $x$ metres be the depth of water at the particular location, $x=5+4\cos\left(\frac{4\pi}{25}t\right)$.

Q13aii Let the first high tide tomorrow is at $t=0$ (2 am).

The tide increases at the fastest rate at $t=\frac{3}{4}\times\frac{25}{2}=\frac{75}{8}$

i.e. 9 hours 22 min 30 s after 2 am

.: The earliest time is 11:22:30 am.

Q13bi When $t=0$, $y=0$

At the highest point, $\dot{y}=0$, $u\sin\theta-10t=0$, $t=\frac{u\sin\theta}{10}$,

$y=u\left(\frac{u\sin\theta}{10}\right)\sin\theta-5\left(\frac{u\sin\theta}{10}\right)^2=\frac{u^2\sin^2\theta}{20}$ above the point of projection.

Q13bii $y=20+\frac{30^2\sin^2 30°}{20}=\frac{125}{4}$ m above the ground

Q13biii Consider the vertical motion, assume $a=10$.

$\frac{1}{2}\times 10\times t^2=\frac{125}{4}$, $t=\frac{5}{2}$ s. It takes 2.5 seconds to reach the ground.

Q13biv Distance from the wall $=10\times 2.5=25$ m

Q13ci $AB$ is a diameter, .: $\angle ACB=\angle ADB=90°$

.: $\angle MCE=\angle MDE=90°$, .: $ME$ is a diameter of the circle $CMDE$

.: quadrilateral $CMDE$ is cyclic.

Q13cii In $\Delta MAB$, $AD$ and $BC$ are altitudes. $MF$ passes through the intersection $E$ of the two altitudes. .: $MF$ must also be an altitude, since the altitudes of a triangle are concurrent. .: $MF$ is perpendicular to $AB$.

Q14ai $4n^3+18n^2+23n+9$

$=(4n^3+14n^2+9n)+(4n^2+14n+9)$

$=n(4n^2+14n+9)+1(4n^2+14n+9)$

$=(n+1)(4n^2+14n+9)$

Q14aii $1\times 3+3\times 5+5\times 7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$

For $n=1$,

$(2\times 1-1)(2\times 1+1)=\frac{1}{3}\times 1\times(4\times 1^2+6\times 1-1)$, i.e. $3=3$, it is true

Assume that for $n=k$,

$1\times 3+3\times 5+5\times 7+\cdots+(2k-1)(2k+1)=\frac{1}{3}k(4k^2+6k-1)$

For $n=k+1$,

Left $=\frac{1}{3}k(4k^2+6k-1)+(2k+1)(2k+3)$

$=\frac{1}{3}k(4k^2+6k-1)+4k^2+8k+3$

$=\frac{4}{3}k^3+6k^2+\frac{23}{3}k+3$

Right $=\frac{1}{3}(k+1)(4(k+1)^2+6(k+1)-1)=\frac{4}{3}k^3+6k^2+\frac{23}{3}k+3$

.: it is true for $n=k+1$

.: it is true for $n\ge 1$

Q14bi $(1+x)^n=\binom{n}{0}x^0+\binom{n}{1}x^1+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n$

Let $x=1$, $2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$

Q14bii

$\frac{d}{dx}(1+x)^n=\frac{d}{dx}\left(\binom{n}{0}x^0+\binom{n}{1}x^1+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n\right)$

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+4\binom{n}{4}x^3+\cdots+n\binom{n}{n}x^{n-1}$

Let $x=1$, $n2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+4\binom{n}{4}+\cdots+n\binom{n}{n}$

Q14biii $n2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+4\binom{n}{4}+\cdots+n\binom{n}{n}$

.: $n2^{n-1}\times 2=n2^n=2\binom{n}{1}+4\binom{n}{2}+6\binom{n}{3}+\cdots+2r\binom{n}{r}+\cdots+2n\binom{n}{n}$

Also $2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$

.: $n2^n=n+n\binom{n}{1}+n\binom{n}{2}+n\binom{n}{3}+\cdots+n\binom{n}{r}+n\binom{n}{n}$

.: $n+n\binom{n}{1}+n\binom{n}{2}+\cdots+n\binom{n}{r}+\cdots+n\binom{n}{n}$

$=2\binom{n}{1}+4\binom{n}{2}+6\binom{n}{3}+\cdots+2r\binom{n}{r}+\cdots+2n\binom{n}{n}$

.: $n=\sum_{r=1}^{n}\binom{n}{r}(2r-n)$

Q14ci Tangent at $T(2at, at^2)$: $y=tx-at^2$ ; directrix: $y=-a$

Intersection $D$ : $tx-at^2=-a$ , $x=at-\frac{a}{t}$ , .: $D\left(at-\frac{a}{t},-a\right)$

Q14cii Normal at $T(2at, at^2)$: $x+ty=at^3+2at$

Intersection $R$ : $x=at-\frac{a}{t}$, $at-\frac{a}{t}+ty=at^3+2at$

.: $y=a\left(t^2+\frac{1}{t^2}+1\right)$ and $x=a\left(t-\frac{1}{t}\right)$

$x^2=a^2\left(t^2+\frac{1}{t^2}-2\right)=a^2\left(t^2+\frac{1}{t^2}+1-3\right)=a^2\left(\frac{y}{a}-3\right)$

.: $x^2=ay-3a^2$ , the locus of $R$ lies on a parabola.

Q14ciii Focal length $=\frac{a}{4}$

Q14civ The shortest distance between $R$ and $T$ occurs when $RT$ is a common normal to the two parabolas.

Normal at $T(2at, at^2)$: $x+ty=at^3+2at$ , $y=-\frac{1}{t}x+at^2+2a$

Normal at $R\left(a\left(t-\frac{1}{t}\right), a\left(t^2+\frac{1}{t^2}+1\right)\right)$ to $x^2=ay-3a^2$ :

Gradient of tangent $\frac{dy}{dx}=\frac{2x}{a}$,

gradient of normal $=-\frac{a}{2x}=-\frac{a}{2a\left(t-\frac{1}{t}\right)}=-\frac{1}{2\left(t-\frac{1}{t}\right)}$

Let $-\frac{1}{t}=-\frac{1}{2\left(t-\frac{1}{t}\right)}$ , .: $2\left(t-\frac{1}{t}\right)=t$ , .: $t=\pm\sqrt{2}$

*Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.*