2016 VCAA Mathematical Methods Sample (v2 April) Exam 2 Solutions

CAS should be used whenever possible to speed up the solution process.

SECTION A

1	2	3	4	5	6	7	8	9	10
A	D	C	E	A	A	C	E	E	C

11	12	13	14	15	16	17	18	19	20
B	B	A	B	D	E	D	B	D	E

Q1 A

Q2 D

Q3 $T = \frac{\pi}{2\pi} = \frac{1}{2}$ C

Q4 $P(-a) = 7(-a)^3 + 9(-a)^2 - 5a(-a) = 0$

$-7a^3 + 9a^2 + 5a^2 = 0$, $a^2(-7a + 14) = 0$, $a = 2$ E

Q5 $-\frac{\pi}{2} \le 2\left(x - \frac{\pi}{6}\right) \le \frac{\pi}{2}$, $-\frac{\pi}{4} \le x - \frac{\pi}{6} \le \frac{\pi}{4}$, $-\frac{\pi}{12} \le x \le \frac{5\pi}{4}$

$\therefore$ $a = \frac{\pi}{12}$ A

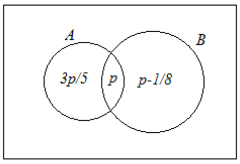
Q6 $\int_1^4 (5 - 2f(x))\,dx = \int_1^4 5\,dx - 2\int_1^4 f(x)\,dx = [5x]_1^4 - 2\times 6 = 3$ A

Q7 $\Pr(A) = \frac{3p}{5} + p = \frac{8p}{5}$ C

$\Pr(B) = p + p - \frac{1}{8} = 2p - \frac{1}{8}$, $\therefore$ $p \ne 0$

$\Pr(A \cap B) = \Pr(A)\Pr(B)$, independent

$\therefore$ $p = \frac{8p}{5}\left(2p - \frac{1}{8}\right)$, $\therefore$ $p = \frac{3}{8}$



Q8 $f(f(x)) = x$, $\therefore$ $f^{-1}(x) = f(x)$ E

Q9 $\Pr(\text{different colours}) = \frac{{}^5C_1 \times {}^4C_1}{{}^9C_2} = \frac{5}{9}$ E

Q10 $x = 1 - x'$, $2y = 2 + y'$, $x - 2y = 3$

$\therefore$ $1 - x' - (2 + y') = 3$, $\therefore$ $-x' - y' = 4$ C

Q11 At $x = c$, $y = e^{ax} = e^{ac}$, $\frac{dy}{dx} = ae^{ax} = ae^{ac}$

Gradient of tangent $= \frac{e^{ac}}{c} = ae^{ac}$, $\therefore$ $c = \frac{1}{a}$ B

Q12 $y = \frac{a}{3}x - \frac{5}{3}$ and $y = \frac{3}{a}x - \frac{8-a}{a}$

No solution: $\frac{a}{3} = \frac{3}{a}$ and $\frac{5}{3} \ne \frac{8-a}{a}$, $\therefore$ $a = -3$ B

Q13 Binomial: $\Pr(X > 5) = 1 - \Pr(X \le 5) \approx 0.0239$ A

Q14 $\Pr(-1.65 < Z < 1.65) \approx 0.90$ B

Q15 $f(x) = ax^3 - bx^2 + cx$, $f'(x) = 3ax^2 - 2bx + c$

$f(x)$ has no stationary points when $3ax^2 - 2bx + c > 0$, i.e. when its $\Delta < 0$, $(-2b)^2 - 4(3a)c < 0$, $3ac > b^2$, $c > \frac{b^2}{3a}$ D

Q16 $\int_0^a (x^2 - 4)\,dx = 0$, $\left[\frac{x^3}{3} - 4x\right]_0^a = 0$, $a = 2\sqrt{3}$ E

Q17 Consider $f(x) = x^3 - 9x^2 + 15x$.

$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$

Local maximum $(1, 7)$, local minimum $(5, -25)$

Translate $f(x) = x^3 - 9x^2 + 15x$ downwards more than 7 units, or upwards more than 25 units. $\therefore$ $w < -7$ or $w > 25$ D

Q18 Stationary points are at $x = -3$, 2.

$f'(x)$ has a maximum, $\therefore$ $y = f'(x)$ is an inverted parabola

$\therefore$ $y = f(x)$ has a local maximum at $x = 2$ B

Q19 Normal: $\Pr(T < 90) \approx \frac{150}{2000}$, $\Pr\left(Z < \frac{90-120}{\sigma}\right) \approx \frac{150}{2000}$

$\frac{90-120}{\sigma} \approx -1.4395$, $\sigma \approx 20.8$ D

Q20 Let $g(x) = -|x|$ and $f(x) = x(x-1) = x^2 - x$.

$f(g(x)) = x^2 + |x| = \begin{cases} x^2 + x & \text{for } x \ge 0 \\ x^2 - x & \text{for } x < 0 \end{cases}$ E

SECTION B

Q1a Period $= \frac{2\pi}{\frac{\pi}{3}} = 6$, amplitude $= 400$

Q1b

Max. pop. $= 1200 + 400 = 1600$, min. pop. $= 1200 - 400 = 800$

Q1c $n(10) = 1200 + 400\cos\frac{10\pi}{3} = 1000$

Q1d $n(t) < n(10)$, when $2 < t < 4$ and $8 < t < 10$,

fraction of time $= \frac{4}{12} = \frac{1}{3}$

Q2a TSA $=6480$, $2\left(hx+\frac{5hx}{2}+\frac{5x^2}{2}\right)=6480$, $h=\frac{6480-5x^2}{7x}$

Q2b $V(x)=\frac{5x(6480-5x^2)}{14}>0$, $\frac{25x(1296-x^2)}{14}>0$,
$x(1296-x^2)>0$, $x(x-36)(x+36)>0$ when $0<x<36$

Q2c $V(x)=\frac{16200x}{7}-\frac{25x^3}{14}$, $\frac{dV}{dx}=-\frac{75}{14}x^2+\frac{16200}{7}$

Q2d Let $\frac{dV}{dx}=-\frac{75}{14}x^2+\frac{16200}{7}=0$, $x=\sqrt{432}=12\sqrt{3}$ and
$h=\frac{6480-5\times432}{7\times12\sqrt{3}}=\frac{360}{7\sqrt{3}}=\frac{120\sqrt{3}}{7}$ for maximum volume

Q3ai Binomial: $n=20$, $p=\frac{5}{8}$, $\Pr(X\ge10)\approx0.9153$

Q3aii $\Pr(X\ge15\mid X\ge10)=\frac{\Pr(X\ge15)}{\Pr(X\ge10)}\approx\frac{0.1788}{0.9153}\approx0.195$

Q3aiii $\mathrm{E}(\hat{P})=p=\frac{5}{8}$, $\mathrm{Var}(\hat{P})=\frac{p(1-p)}{n}=\frac{\frac{5}{8}\times\frac{3}{8}}{20}=\frac{3}{256}$

Q3aiv $\sigma=\sqrt{\frac{3}{256}}=\frac{\sqrt{3}}{16}$
$\frac{5}{8}-2\times\frac{\sqrt{3}}{16}\approx0.4085$, $\frac{5}{8}+2\times\frac{\sqrt{3}}{16}\approx0.8415$
$\Pr(0.4085<\hat{P}<0.8415)=\Pr(20\times0.4085<X<20\times0.8415)$
$=\Pr(9\le X\le16)\approx0.939$

Q3av $\Pr\left(\hat{P}\ge\frac{3}{4}\mid\hat{P}\ge\frac{5}{8}\right)=\Pr(X\ge15\mid X\ge12.5)=\frac{\Pr(X\ge15)}{\Pr(X\ge13)}$
$\approx\frac{0.1788}{0.5079}\approx0.352$

Q3b $\Pr(FFF')+\Pr(FF'F)+\Pr(F'FF)$
$=\frac{1}{2}\times\frac{3}{4}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}=\frac{11}{32}$

Q3ci $\mathrm{E}(W)=\int_1^3\frac{w((w-3)^3+64)}{256}\,dw+\int_3^5\frac{w(w+29)}{128}\,dw$
$\approx0.978125+2.0677083\approx3.0458$ min

Q3cii $\Pr(W>4)=\int_4^5\frac{w+29}{128}\,dw\approx0.261719$
Expected number of members $\approx200\times0.261719\approx52$

Q3d $\left(0.6-1.96\sqrt{\frac{0.6\times0.4}{100}},\ 0.6+1.96\sqrt{\frac{0.6\times0.4}{100}}\right)\approx(0.504,\ 0.696)$

Q4ai $\int_{-2}^{0}e^x\,dx=\left[e^x\right]_{-2}^{0}=1-e^{-2}$

Q4aii $1-e^{-2}$

Q4aiii Area of shaded region $=\int_{-2}^{1}e^x\,dx=\left[e^x\right]_{-2}^{1}=e-e^{-2}$

Q4bi Let $\log_e(x)=-\log_e(a-x)$, $\log_e(x)+\log_e(a-x)=0$
$\log_e(x)(a-x)=0$, $x(a-x)=1$, $x^2-ax+1=0$, $x=\frac{a\pm\sqrt{a^2-4}}{2}$

Q4bii $a^2-4>0$, given $a>0$ $\therefore$ $a>2$

Q4c $x=\frac{a}{2}=\sqrt{2}$, $a=2\sqrt{2}$

Q5a
$x^4-8x=x(x^3-2^3)=x(x-2)(x^2+2x+4)=x(x-2)((x+1)^2+3)$

Q5b $g(x)=f(x+1)$, i.e. translate $y=f(x)$ in the negative x direction by 1 unit.

Q5ci $1\le d<3$

Q5cii $d<1$

Q5d $y=g(x)=x^4-8x$, $g'(x)=4x^3-8=0$, $x=\sqrt[3]{2}$
$y=(\sqrt[3]{2})^4-8(\sqrt[3]{2})$, $\therefore$ $n=(\sqrt[3]{2})^4-8(\sqrt[3]{2})=-6(\sqrt[3]{2})$

Q5ei $y=g(x)=x^4-8x$, $g'(x)=4x^3-8$
$g'(u)=4u^3-8=m$, $g'(v)=4v^3-8=-m$
$\therefore$ $4u^3-8+4v^3-8=0$, $u^3+v^3=4$

Q5eii $u^3+v^3=(u+v)^3-3uv(u+v)=4$ and given $u+v=1$
$\therefore$ $1-3uv=4$, $uv=-1$. Since $g'(u)=4u^3-8=m>0$,
$\therefore$ $u=\frac{1+\sqrt{5}}{2}$ and $v=\frac{1-\sqrt{5}}{2}$

Q5fi $y=g(x)=x^4-8x$, $g'(x)=4x^3-8$
At $x=p$, $y=g(p)=p^4-8p$, $g'(p)=4p^3-8$
Equation of the tangent: $y-(p^4-8p)=(4p^3-8)(x-p)$
$y=(4p^3-8)x-4p^4+8p+p^4-8p$
$\therefore$ $y=(4p^3-8)x-3p^4$

Q5fii $y=(4p^3-8)x-3p^4$ passes through $\left(\frac{3}{2},-12\right)$
$\therefore$ $-12=(4p^3-8)\frac{3}{2}-3p^4$, $-12=6p^3-12-3p^4$, $6p^3-3p^4=0$
$\therefore$ $3p^3(2-p)=0$, $\therefore$ $p=0$ or 2
Equations of the tangents are:
$y=-8x$ and $y=24x-48$

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