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2017 Mathematical Methods

Year 12 Modelling Task

Time allowed: 2 hours plus

You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator
Working must be shown for questions worth 2 or more marks. Total: 80 marks

Modelling Task

Theme: Approximate probability distributions in sampling

Part I

Consider a population of size N consisting of R individuals of a certain attribute.
A random sample of size n is selected without replacement and there are x individuals of the certain attribute in the sample. x is the value of random variable X .
Random variable X in sampling without replacement has a discrete probability distribution called the ***hypergeometric distribution****.*
If the population is very large and the number of individuals of the certain attribute is proportionally large, then the discrete probability distribution for sampling with replacement, called the ***binomial distribution****, can be used as an approximation to the hypergeometric distribution.*

A year 12 class consists of 25 students. 40% of students in the class are **not** wearing proper school uniform.
Let W represent wearing proper uniform, and N not wearing proper uniform.
Let the number of students **not** wearing proper uniform be random variable *X*.

Question 1 A random sample of 3 students from the class is called to the principal's office together for uniform inspection.

a. How many possible outcomes of the uniform inspection by the principal? 1 mark

b. Draw a tree diagram showing all possible outcomes and corresponding probabilities. 3 marks

c. Hence complete the following probability distribution of the uniform inspection. 3 marks

X				
$\Pr(X = x)$				

d. Complete the following table.
Use combinatorics instead of the tree diagram to compute the probabilities.
Write your answers in the same form as the example shown in the table. 3 marks

X			2	
$\Pr(X = x)$			$\frac{{}^{10}C_2 \times {}^{15}C_1}{{}^{25}C_3}$	

e. Write a probability function including its domain for random variable *X*. 2 marks

f. Calculate the expected value of *X*. 2 marks

Question 2 The principal decides to change the selection process for uniform inspection. A randomly selected student is called to the office and allowed to return to class before the next student (also selected randomly from the full class including the returned student) is called.

a. Complete the following probability distribution of the school uniform inspection. 3 marks

X				
$\Pr(X = x)$				

b. Write a probability function including its domain for random variable *X*. 2 marks

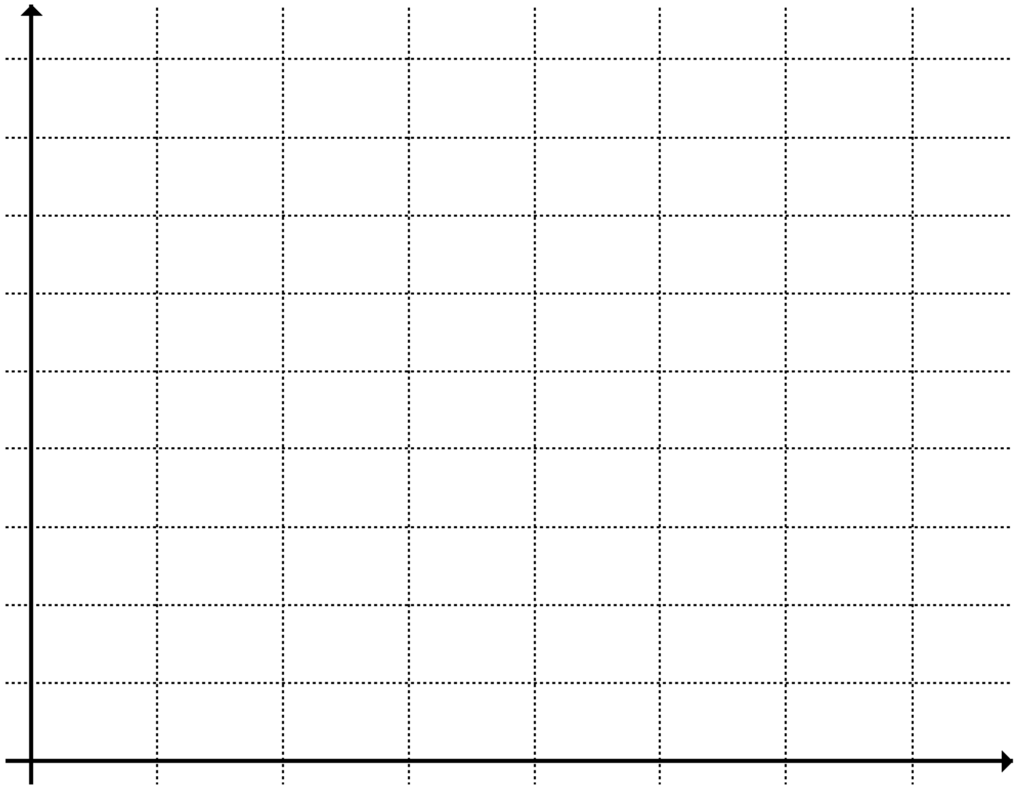
c. Write down the expected value of *X*. 1 mark

d. Write down the standard deviation of *X*. 1 mark

Question 3

a. Plot the graphs of the probability distributions obtained in **Questions 1 and 2** on the same set of axes. Use $\bullet$ for Question 1, and $*$ for Question 2.

4 marks



b. Comment on the closeness of the graph of Question 2 to the graph of Question 1. 2 marks

The year 12 level at the school consists of 100 students. 40% of year 12 students are **not** wearing proper school uniform.
Let the number of students **not** wearing proper uniform be random variable *X*.

Question 4 A random sample of 3 students from the year 12 level is called to the principal's office together for uniform inspection. Complete the following table. Use CAS to compute the probability for each value of *X*.

2 marks

X				
$\Pr(X = x)$				

Question 5 One student selected randomly is called to the office and allowed to return to class before the next student (also selected randomly from the whole year 12 level including the returned student) is called. Three students in total were called.
Complete the following table. Use CAS to compute the probability for each value of *X*.

3 marks

X				
$\Pr(X = x)$				

Question 6 Comment on the closeness of the two probability distributions of random variable *X* obtained in **Questions 4 and 5**.

2 marks

Question 7 The school has a student population of N. The proportion of the student population not wearing proper school uniform is p.
In one uniform inspection, 3 students are randomly selected together, i.e. without replacement. In another inspection, 3 students are randomly selected one at a time with replacement.

a. In terms of N and p, how many students are not wearing proper uniform? 1 mark

b. Using sampling **with** replacement method find $\Pr(X=1)$ in terms of p and/or N. 1 mark

c. Using sampling **without** replacement method find $\Pr(X=1)$ in terms of C, p and N. 1 mark

d. Write an equivalent expression for $\Pr(X=1)$ in part c in terms of p and N only. 1 mark

e. Hence show that the answer in part c approaches the answer in part b as N becomes very large.
Hint: Divide the numerator and denominator by N^3, rearrange the expression and let $N \to \infty$.
2 marks

Part II

If the sample is large (i.e. n is large) and the binomial distribution is close to symmetric (i.e. p is close to 0.5), then a continuous probability distribution called the ***normal distribution*** *is a good approximation, using the same $\mu = np$ and $\sigma = \sqrt{np(1-p)}$ as in the binomial distribution.*
Since the normal distribution is a continuous distribution, $\Pr(X = x)$ in the binomial distribution corresponds to $\Pr(x - 0.5 < X < x + 0.5)$ in the normal distribution.

The year 12 level at the school consists of 100 students. 40% of the year 12 students are **not** wearing proper school uniform.
Let the number of students **not** wearing proper uniform be random variable *X*.

Question 8 Investigate the closeness of the normal distribution to the binomial distribution for a random sample of 3 students from the year 12 level called to the principal's office together for uniform inspection. In this case $n = 3$ is considered small.

a. Calculate μ and σ of X in the normal distribution to be used as an approximation to the binomial distribution. 2 marks

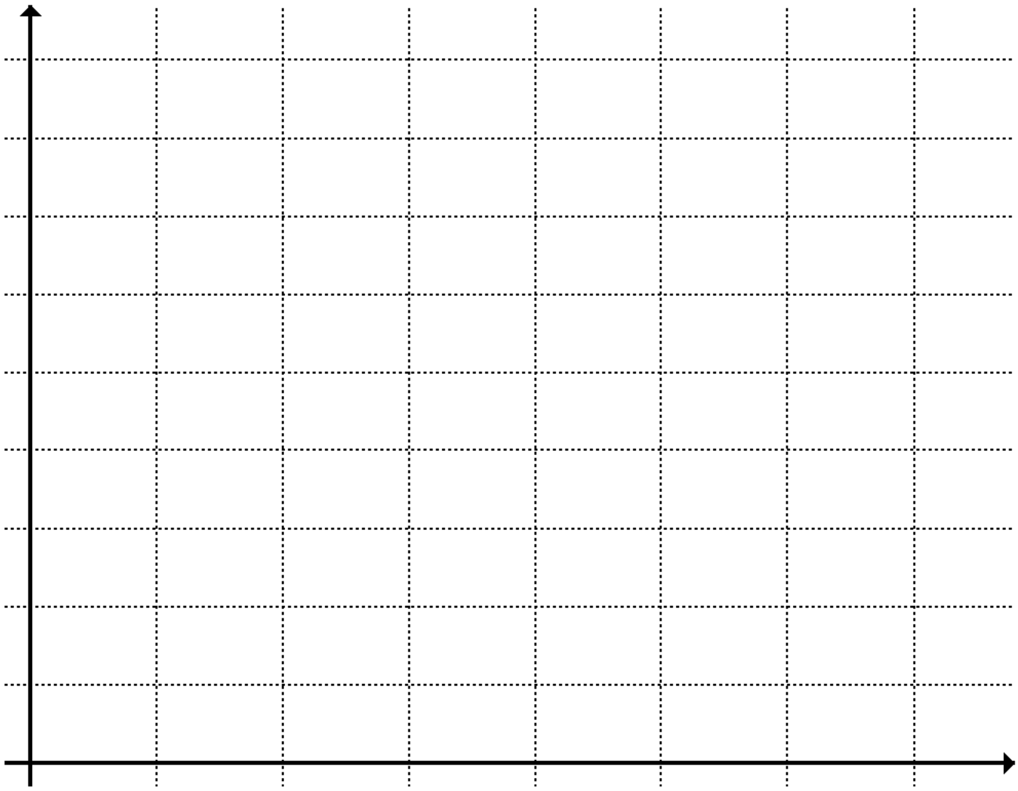
b. Complete the following table. Use CAS to compute the probability for each value of *X*. 3 marks

X				
Binomial: $\Pr(X = x)$				
Normal: $\Pr(x - 0.5 < X < x + 0.5)$				

c. Comment on the sum of the normal probabilities in the above table. 2 marks

d. Plot the binomial and normal data on the same set of axes. Use • for the binomial data, and * for the normal data.

3 marks



e. Comment on the closeness of the two distributions in terms of their shapes and probability values of X .

2 marks

Question 9 Investigate the closeness of the normal distribution to the binomial distribution for a random sample of 6 students from the year 12 level called to the principal's office together for uniform inspection.

a. Calculate μ and σ of X in the normal distribution to be used as an approximation to the binomial distribution.

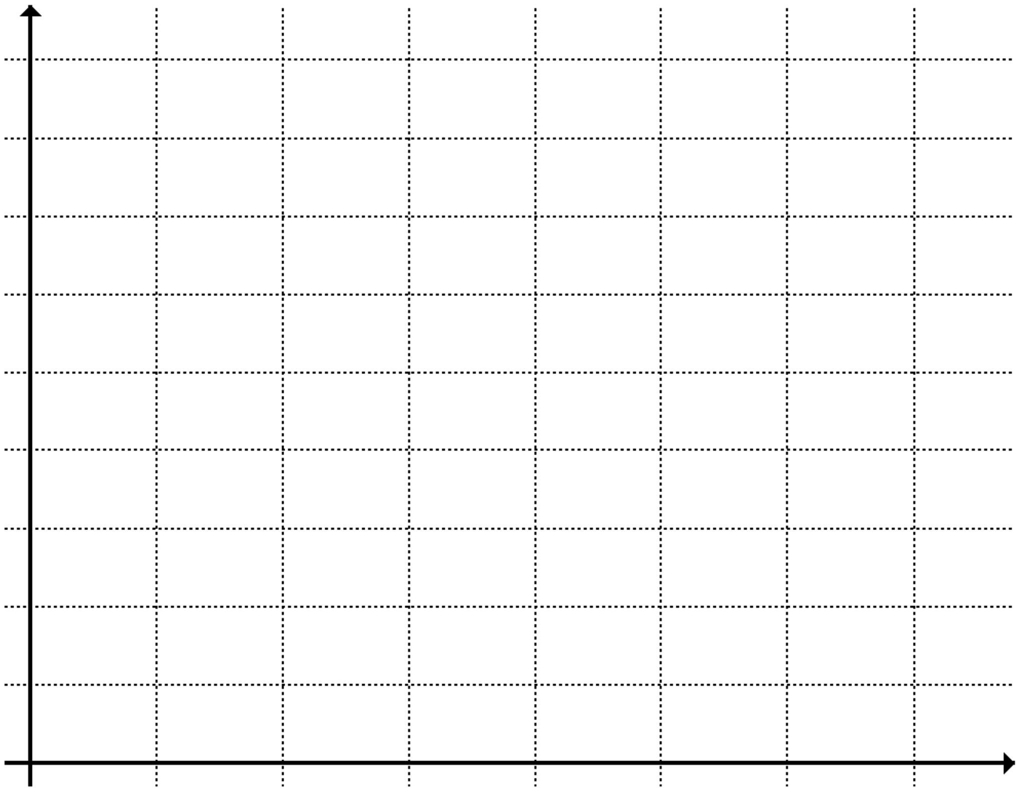
1 mark

b. Complete the following table. Use CAS to compute the probability for each value of X. 4 marks

X								
Binomial: $\Pr(X = x)$								
Normal: $\Pr(x - 0.5 < X < x + 0.5)$								

c. Plot the binomial and normal data on the same set of axes. Use • for the binomial data, and $*$ for the normal data.

3 marks



d. Discuss the closeness of the two distributions when the value of n changes. Use the graphs to justify your comments.

2 marks

Question 10 The school uniform issue has worsened. 50% of the year 12 students are now **not** wearing proper school uniform.
A random sample of **3** students from the year 12 level is called to the principal's office together for uniform inspection.
Going through the same analysis as that done in Question 8, investigate the effects of changing p to 0.50 (i.e. 50%) on the closeness of the normal distribution to the binomial distribution.

8 marks

Consider another random variable Y called the sample proportion. It is defined as $Y = \frac{X}{n}$, where X is the number of students in the sample **not** wearing proper uniform, and n (a constant) is the size of the sample.

Question 11

a. Show that $\mathrm{E}(Y) = p$ where p is the proportion of students in the year 12 level **not** wearing proper uniform.

1 mark

b. Show that $\sigma_Y = \sqrt{\frac{p - p^2}{n}}$.

2 marks

c. Use part a and /or part b to discuss or explain the effects on Y when the sample size is very large.

2 marks

d. Given $n = 20$ and $p = 0.40$ (40%), find $\Pr(Y = 0.40)$ correct to 2 decimal places.

2 marks

e. Given $n = 20$, $Y = 0.40$ and the value of p is not known, find the 95% confidence interval for p (use $z = 2$) correct to 2 decimal places.

3 marks

End of task