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# 2017 Mathematical Methods

# Year 12
## Problem Solving Task

*Time allowed: 2 hours plus*

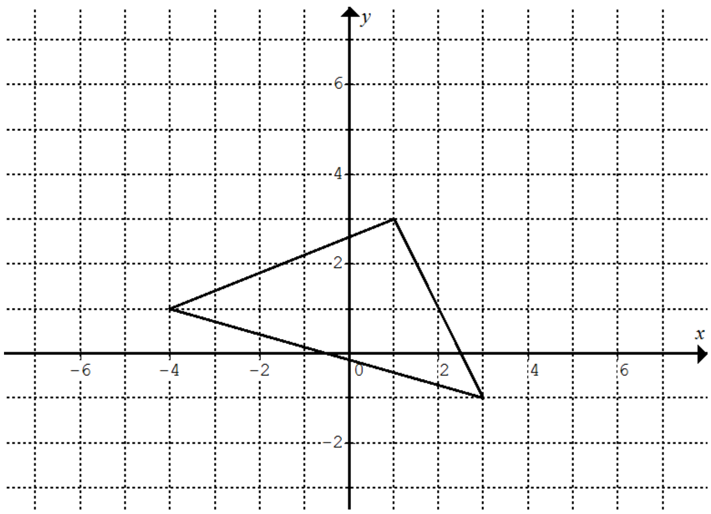
**You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator**
***Working must be shown for questions worth 2 or more marks. Total: 70 marks***

**Theme: Shapes bounded by a parabola**

If a figure is dilated by the same factor in both $x$ and $y$ directions, the resulting figure is similar to the original figure.
Translations and reflections in the $x$ and $y$ axes do not change the similarity of parabolas.
All parabolas are similar.

**Problem 1**

The triangle shown in the following diagram is dilated by a factor of $1.5$ in both $x$ and $y$ directions.




a. Sketch the resulting figure on the same diagram. 2 marks

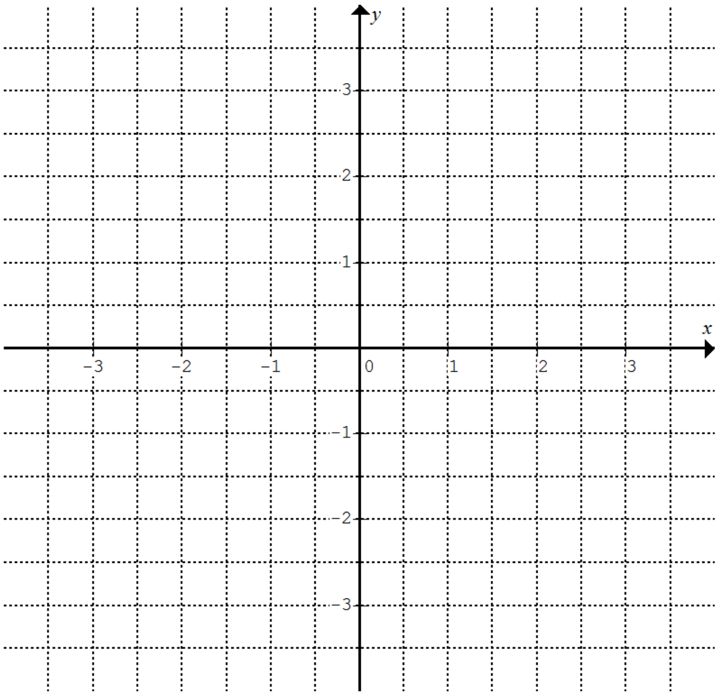
b. Show that the resulting figure is similar to the given figure by calculating and comparing the side lengths of the two figures.
4 marks

**Problem 2**

Consider the parabola (labeled as P) given by the equation $y = 1 - x^2$.
A similar parabola is formed when $y = 1 - x^2$ is dilated by a factor of 2 in both $x$ and $y$ directions.

a. The equation of the resulting figure (labeled as Q) can be expressed in the form $y = a - bx^2$.
Find the value of each of $a$ and $b$. 3 marks

b. Sketch the graphs of P and Q on the diagram below. Show coordinates of axis intercepts. 3 marks




Another parabola (labeled as R) is formed when $y = 1 - x^2$ is dilated by a factor of 2 in the $x$ direction only.

c i. Find the equation of R. 1 mark

c ii. Show that R is also the result of dilations of $y = \frac{1}{4} - x^2$ by a factor of 4 in both $x$ and $y$ directions.

2 marks

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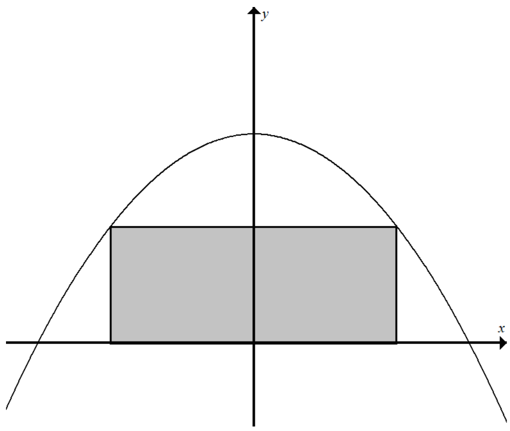
c iii. Explain clearly why parabola R is similar to P. 2 marks

**Problem 3**

Using the ideas in Problem 1 and Problem 2 to explain clearly why $y = \sqrt{2}\, x^2$ is similar to $y = 1 - x^2$, knowing that reflections in the $x$ and $y$ axes do not change the similarity of parabolas.

3 marks

**Problem 4**




The diagram above shows a rectangle enclosed by the parabola $y = 1 - x^2$ and the $x$ axis.
Let $\ell$ be the length of the base of the rectangle.

a. Find the height of the rectangle and its area in terms of $\ell$. 2 marks

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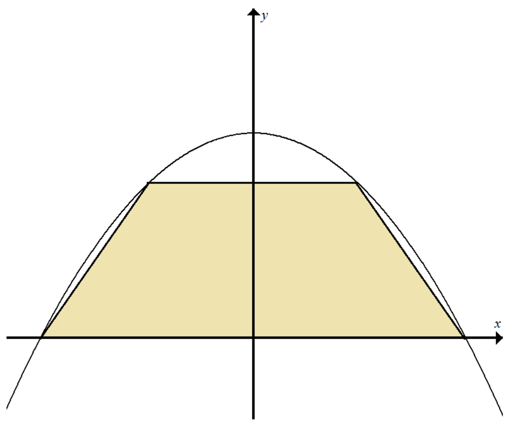
b. Find the exact value of the maximum area of the rectangle without the use of CAS. 3 marks

**Problem 5**

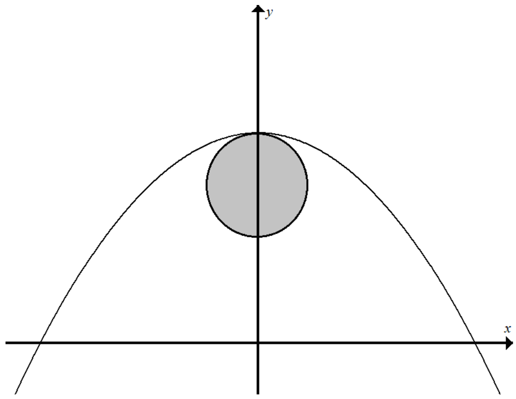



The diagram above shows a trapezium enclosed by the parabola $y = 1 - x^2$ and the $x$ axis.
Let $h$ be the height of the trapezium.

a. Find the area of the trapezium in terms of $h$. 2 marks

b. Find the exact value of the maximum area of the trapezium without the use of CAS. 3 marks

**Problem 6**




The diagram above shows a circle of radius $r$ below the parabola $y = 1 - x^2$ and touching it at its vertex.

a. Determine the equation of the circle in terms of $r$. 3 marks

b. Write an equation including its domain in terms of $r$ for the top half of the circle. 1 mark

For the circle below the parabola $y = 1 - x^2$ and touching it at its vertex only, it has to be at or lower than the parabola in the domain.

c. Find the exact radius of the largest circle below the parabola $y = 1 - x^2$ and touching it at its vertex only. Hence find the exact area of the largest circle possible.

4 marks

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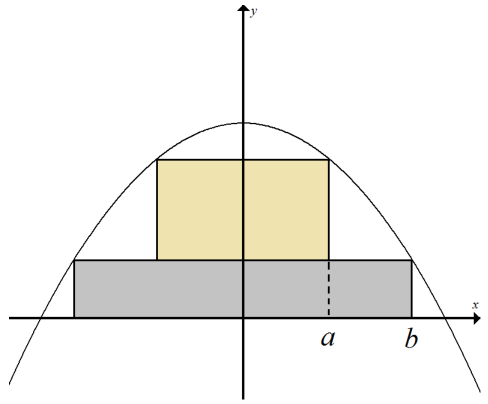
**Problem 7**




The diagram above shows two stacked rectangles enclosed by the parabola $y = 1 - x^2$ and the $x$ axis.
The top rectangle has a base length of $2a$, and the bottom rectangle has a base length of $2b$.
The problem is to find the maximum total area of the two rectangles.

a. Find the height of the top rectangle in terms of $a$ and $b$. 2 marks

b. Let $A$ be the total area of the two rectangles. Show that $\frac{A}{2} = ab^2 + 2b - a^3 - b^3$. 2 marks

c. Write in each blank cell in the table the value of $\frac{A}{2}$ for the corresponding values of $a$ and $b$. 6 marks

| $b$ \ $a$ | 0.400 | 0.405 | 0.410 | 0.415 | 0.420 | 0.425 | 0.430 | 0.435 | 0.440 |
|---|---|---|---|---|---|---|---|---|---|
| 0.700 | | | | | | | | | |
| 0.705 | | | | | | | | | |
| 0.710 | | | | | | | | | |
| 0.715 | | | | | | | | | |
| 0.720 | | | | | | | | | |
| 0.725 | | | | | | | | | |
| 0.730 | | | | | | | | | |
| 0.735 | | | | | | | | | |
| 0.740 | | | | | | | | | |
| 0.745 | | | | | | | | | |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d. Use the completed table above to estimate the maximum total area of the two rectangles. 1 mark

e. Estimate the contribution to the maximum total area from each rectangle. 2 marks

**Problem 8** Here you use a calculus approach to find the maximum total area $A$ of the two rectangles in Problem 7.

In Problem 7 part b you have shown $\frac{A}{2} = ab^2 + 2b - a^3 - b^3$.

a. By considering $b$ as a constant, find $\frac{d}{da}\left(\frac{A}{2}\right)$. 1 mark

b. By considering $a$ as a constant, find $\frac{d}{db}\left(\frac{A}{2}\right)$. 1 mark

To maximise $A$, the values of $a$ and $b$ must satisfy simultaneously $\frac{d}{da}\left(\frac{A}{2}\right) = 0$ and $\frac{d}{db}\left(\frac{A}{2}\right) = 0$.

c. Write down the two simultaneous equations in terms of variables $a$ and $b$. 1 mark

d. Show that $b = \sqrt{3}\,a$ in order to maximise $A$. 1 mark

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e. Hence show algebraically that $a^2 = \frac{1}{9-2\sqrt{3}}$ in order to maximise $A$. 2 marks

f. Hence show algebraically $\left(\frac{A_{\max}}{2}\right)^2 = \frac{9+2\sqrt{3}}{207}$, where $A_{\max}$ is the maximum area of the two rectangles. Full working is required. Do not use CAS. 6 marks

**Problem 9** The maximum areas of the shapes in Problem 4, Problem 5, Problem 6 and Problem 8 are to be determined using parabola $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ to replace parabola $y = 1 - x^2$.

a. Explain/show that $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ is similar to $y = 1 - x^2$. State the dilation factor used in both *x* and *y* directions. 2 marks

Hence or otherwise,

b i. find the exact maximum area of the rectangle in Problem 4 enclosed by $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ and the $x$ axis.

1 mark

b ii. find the exact maximum area of the trapezium in Problem 5 enclosed by $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ and the $x$ axis.

1 mark

b iii. find the exact maximum area of the circle in Problem 6 when parabola $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ is used.

1 mark

b iv. find the exact maximum total area of the stacked rectangles in Problem 8 enclosed by $y = \sqrt{3} - \frac{x^2}{\sqrt{3}}$ and the $x$ axis.

2 marks

**END OF TASK**

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