2017 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	B	B	C	D	D	A	C	C	B

Q1 $f(2)=8-20+22-10=0$, .: $x-2$ is a factor. **A**

Q2 $\log_a 8=\log_a 4^{\frac{3}{2}}=\frac{3}{2}\log_a 4$

.: $\log_a 4=\frac{2}{3}\log_a 8=\frac{2}{3}\times 1.893=1.262$ **B**

Q3 $\angle TAD=\angle TAB+\angle BAD=65^\circ+(180-110)^\circ=135^\circ$ **B**

Q4 $\sqrt{5}\sin(x-\alpha)=\sqrt{5}\sin x\cos\alpha-\sqrt{5}\cos x\sin\alpha=2\sin x-\cos x$

.: $\sqrt{5}\sin\alpha=1$ and $\sqrt{5}\cos\alpha=2$, .: $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{1}{2}$ **C**

Q5 $y=\frac{2x^2}{1-x^2}=-\frac{2}{(x-1)(x+1)}-2$ **D**

Q6 $x^2=4y$, $a=1$ At $(2t,t^2)$, normal: $x+ty=t^3+2t$

At $\left(\frac{2}{p},\frac{1}{p^2}\right)$, normal: $x+\frac{y}{p}=\frac{1}{p^3}+\frac{2}{p}$, .: $p^2y+p^3x=1+2p^2$ **D**

Q7 $\frac{3}{x}\geq -1$, .: $\frac{x}{3}\leq -1$, .: $x\leq -3$

$\frac{3}{x}\leq 1$, .: $\frac{x}{3}\geq 1$, .: $x\geq 3$ **A**

Q8 $\frac{dr}{dt}=5$, $A=\pi r^2$

$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}=2\pi r\times 5=150\pi$ when $r=15$ **C**

Q9 $(x^3)^4\left(\frac{1}{x^4}\right)^3=1$ **C**

Q10 $\Pr=\frac{(2+3+3)}{\binom{9}{3}}=\frac{2}{21}$ **B**

Section II

Q11a $x=\frac{3(-4)+2(1)}{2+3}=-2$

Q11b $\frac{d}{dx}\tan^{-1}(x^3)=\frac{3x^2}{1+(x^3)^2}=\frac{3x^2}{1+x^6}$

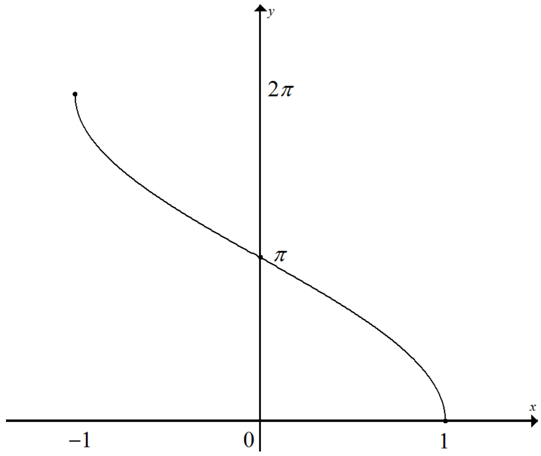
Q11c $\frac{2x}{x+1}>1$, $2-\frac{2}{x+1}>1$, .: $\frac{2}{x+1}<1$

For $x>-1$, $2<x+1$, .: $x>1$

For $x<-1$, $\frac{2}{x+1}<1$ is true.

.: $x<-1$ or $x>1$

Q11d



Q11e $u^2=x+1$, $2\frac{du}{dx}=\frac{1}{u}$

$\int_0^3\frac{x}{\sqrt{x+1}}dx=\int_0^3\frac{u^2-1}{u}dx=\int_0^3 2(u^2-1)\frac{du}{dx}dx=\int_1^2 2(u^2-1)du$

$=\left[\frac{2u^3}{3}-2u\right]_1^2=\frac{8}{3}$

Q11f $u=\sin x$, $\frac{du}{dx}=\cos x$

$\int\sin^2 x\cos x\,dx=\int u^2\frac{du}{dx}dx=\int u^2\,du=\frac{\sin^3 x}{3}+c$

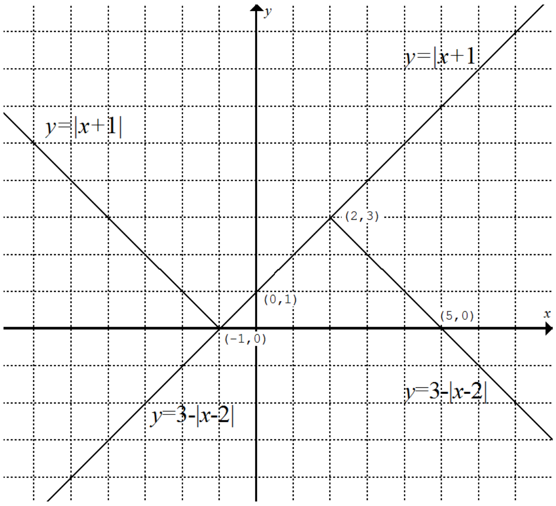
Q11gi $\binom{8}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5$

Q11gii $\binom{8}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^8=\left(\frac{4}{5}\right)^8$

Q11giii $1-\left(\frac{4}{5}\right)^8$

Q12a $\angle ABC = \frac{1}{2}(360° - 100°) = 130°$. The angle subtended on the circumference by an arc equals to a half of the angle subtended at the centre by the same arc.

Q12bi



Q12bii $|x+1|+|x-2|=3$, i.e. $|x+1|=3-|x-2|$, the intersection of the two graphs. .: $-1 \le x \le 2$

Q12ci The piece on the right generates a solid of volume which is a third of the volume of the sphere.

$\int_h^1 \pi\left(\sqrt{1-x^2}\right)^2 dx = \frac{1}{3}\left(\frac{4}{3}\pi \times 1^3\right)$, $\left[x - \frac{x^3}{3}\right]_h^1 = \frac{4}{9}$

.: $\frac{2}{3} - h + \frac{h^3}{3} = \frac{4}{9}$, $3h^3 - 9h + 2 = 0$.

Q12cii Let $f(h) = 3h^3 - 9h + 2$, $f'(h) = 9h^2 - 9$

$h_1 = 0$, $h_2 \approx h_1 - \frac{f(h_1)}{f'(h_1)} = 0 - \frac{2}{-9} = \frac{2}{9}$

Q12d $t = 4 - e^{-2x}$, $\frac{dt}{dx} = 2e^{-2x}$, $\frac{dx}{dt} = \frac{1}{2}e^{2x}$,

$a = \frac{d^2x}{dt^2} = e^{2x}\frac{dx}{dt} = \frac{1}{2}\left(e^{2x}\right)^2 = \frac{1}{2}e^{4x}$

Q12e $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)^2 \to 2(1) = 2$

Q13a $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$, centred at the origin.

.: $v^2 = c - n^2 x^2$

Given $v = 4$ when $x = 2$, and $v = 3$ when $x = 5$

.: $n^2 = \frac{1}{3}$, $n = \frac{1}{\sqrt{3}}$, period $= \frac{2\pi}{n} = 2\pi\sqrt{3}$

Q13bi Given n is a positive even integer.

If k is odd, $x^k + (-x)^k = 0$, if k is even, $x^k + (-x)^k = 2x^k$.

$(1+x)^n + (1-x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=0}^{n}\binom{n}{k}(-x)^k$

$= 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]$

Q13bii $\frac{d}{dx}\left[(1+x)^n + (1-x)^n\right] = \frac{d}{dx}2\left[\binom{n}{0} + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]$

$n\left[(1+x)^{n-1} - (1-x)^{n-1}\right] = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \ldots + n\binom{n}{n}x^{n-1}\right]$

Q13biii Let $x = 1$.

$n\,2^{n-1} = 2\left[2\binom{n}{2} + 4\binom{n}{4} + \ldots + n\binom{n}{n}\right]$

$n\,2^{n-1} = 2^2\left[\binom{n}{2} + 2\binom{n}{4} + \ldots + \frac{n}{2}\binom{n}{n}\right]$

.: $\binom{n}{2} + 2\binom{n}{4} + \ldots + \frac{n}{2}\binom{n}{n} = n\,2^{n-3}$

Q13ci $x = Vt\cos\theta$, $y = Vt\sin\theta - \frac{1}{2}gt^2$

Eliminate t: $y = x\left(\tan\theta - \frac{g}{2V^2\cos^2\theta}x\right)$

Let $y = 0$, $x = 0$ or

$x = \frac{2V^2\cos^2\theta\tan\theta}{g} = \frac{V^2 2\sin\theta\cos\theta}{g} = \frac{V^2\sin 2\theta}{g}$

.: the horizontal range $= \frac{V^2\sin 2\theta}{g} - 0 = \frac{V^2\sin 2\theta}{g}$ metres

Q13cii If $V^2 < 100g$, $\frac{V^2\sin 2\theta}{g} < 100\sin 2\theta$

.: the horizontal range $< 100\sin 2\theta \le 100$ metres

Q13ciii Given $V^2 = 200g$ and the horizontal range ≥ 100

.: $200\sin 2\theta \ge 100$, $\sin 2\theta \ge \frac{1}{2}$, $\frac{\pi}{6} \le 2\theta \le \frac{5\pi}{6}$, .: $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$

Q13civ The max. height occurs at the greatest allowable θ, i.e. $\frac{5\pi}{12}$

Vertical component: $v^2 = u^2 + 2as$, $v = 0$, $u = V\sin\theta$, $a = -g$

.: $s = \frac{-u^2}{2a} = \frac{V^2\sin^2\theta}{2g} = 100\sin^2\frac{5\pi}{12} = 50\left(1 - \cos\frac{5\pi}{6}\right) = 25\left(2+\sqrt{3}\right)$

The greatest height is $25\left(2+\sqrt{3}\right)$ metres.

Q14ai $n=1$, $8^3+6^1=518$ is divisible by 7
$n=k$, assume that $8^{2k+1}+6^{2k-1}$ is divisible by 7
Consider $n=k+1$:
$8^{2(k+1)+1}+6^{2(k+1)-1}=8^2\times 8^{2k+1}+6^2\times 6^{2k-1}$
$=8^2\left(8^{2k+1}+6^{2k-1}\right)-\left(8^2-6^2\right)6^{2k-1}=8^2\left(8^{2k+1}+6^{2k-1}\right)-28\times 6^{2k-1}$
Since 28 is divisible by 7 and $8^{2k+1}+6^{2k-1}$ is assumed to be divisible by 7, .: $8^{2(k+1)+1}+6^{2(k+1)-1}$ is divisible by 7.
.: $8^{2n+1}+6^{2n-1}$ is divisible by 7 for any integer $n\geq 1$.

Q14bi At $P\left(2p,\, p^2\right)$, tangent to $x^2=4y$: $y=px-p^2$
Solve together with $x^2=-4ay$, $x^2=-4a\left(px-p^2\right)$
.: $x^2+4apx-4ap^2=0$

Q14bii $x_M=-\dfrac{4ap}{2}=-2ap$, $y_M=p(-2ap)-p^2=-2ap^2-p^2$

Q14biii $x^2=-4y$, $(-2ap)^2=-4\left(-2ap^2-p^2\right)$
.: $4a^2p^2-8ap^2-4p^2=0$, $4p^2\left(a^2-2a-1\right)=0$
For $p>0$ and $a>0$, $a^2-2a-1=0$, .: $a=\dfrac{2+\sqrt{4+4}}{2}=1+\sqrt{2}$

Q14ci Given $F'(t)=50e^{-0.5t}-0.4F(t)$
$\dfrac{d}{dt}\left(F(t)e^{0.4t}\right)=e^{0.4t}F'(t)+0.4e^{0.4t}F(t)$
$=e^{0.4t}\left(50e^{-0.5t}-0.4F(t)\right)+0.4e^{0.4t}F(t)=50e^{-0.1t}$

Q14cii From part i, $F(t)e^{0.4t}=\int 50e^{-0.1t}dt=-500e^{-0.1t}+c$
.: $F(t)=\dfrac{-500e^{-0.1t}+c}{e^{0.4t}}$
Given $F(0)=0$, .: $c=500$,
.: $F(t)=\dfrac{-500e^{-0.1t}+500}{e^{0.4t}}=500\left(e^{-0.4t}-e^{-0.5t}\right)$

Q14ciii Let $F'(t)=0$, $50e^{-0.5t}=0.4F(t)$, $F(t)=125e^{-0.5t}$
$500\left(e^{-0.4t}-e^{-0.5t}\right)=125e^{-0.5t}$, $4\left(e^{-0.4t}-e^{-0.5t}\right)=e^{-0.5t}$
$4\left(e^{0.1t}-1\right)=1$, $e^{0.1t}=\dfrac{5}{4}$, $t=10\ln\left(\dfrac{5}{4}\right)$

Please inform mathline@itute.com re conceptual and/or mathematical errors.